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Mapping of a General Type of Three Dimensional Region on a Sphere.*

By John J. Gergen.†

Part I. Mapping of the Interior.

1. *Introduction.* We shall be concerned in this paper with a three dimensional region D having the following properties:

1. It is bounded;
2. It is normal,‡ that is, a solution of the classical Dirichlet problem exists for D for any assigned continuous boundary values. In particular, if P_0 be any point of D, then the Green's function $g(D \mid P_0, P)$ with pole at P_0, exists for D;
3. There is a point O in D such that $\nabla g(D \mid O, P)$ vanishes at no point P in D.

We shall prove that this region D can be mapped upon the interior D' of a sphere in such a way that:

1. The images in D' of points on a level surface $g(D \mid O, P) = \text{const.}$ lie on a sphere concentric with D';

* Preliminary report presented to the American Mathematical Society, February, 1928, under the title of "On Accessible Points on the Boundary of a Three Dimensional Region." This problem was suggested by Professor G. C. Evans, and the author is particularly indebted to Professor Evans for suggesting—on the basis of physical intuition of the flow of electricity—a type of simply connected region for which $\mid \nabla g \mid$ vanishes at an interior point, and for the proof of Lemma 7, second paper, which takes the place of two lemmas given by Professor Osgood for the two dimensional situation.

† National Research Fellow.

‡ For a discussion of the methods of attack on the problem of Dirichlet, sufficient conditions for normality, and a complete set of references, see Kellogg, "Recent Progress with the Dirichlet Problem," *Bulletin of the American Mathematical Society*, Vol. 32 (1926), pp. 601-625. See also Kellogg, *Foundations of Potential Theory*, Berlin, (1929). Another reference to the problem at hand is Bouligand, "Sur la Notion Générale de Flux de Green et Son Application à l'Étude du Principe de Picard," *Journal de Mathématiques Pures et Appliquées*, t. 7 (1928), pp. 93-111. In this memoir Bouligand discusses the problem of Dirichlet and the principle of Picard, employing in his analysis the trajectories orthogonal to the level surfaces of the 'generalized' Green's function. He considers likewise in a general way, without aiming at detailed proofs, some of the aspects of our present mapping problem.

2. The images of points on any trajectory orthogonal to these level surfaces lie on a radius of D';
3. The points in D correspond in a 1:1 manner to those in D';
4. The mapping functions (of the coördinates) which carry D into D' are analytic in $D - O$ and continuous at O.

The importance of this type of mapping lies, of course, in its analogy to conformal mapping of plane regions upon a circle. This analogy rests on the fact that when a plane region S is mapped in a 1:1 and conformal manner upon the interior S' of a circle, the level curves of the Green's function of S, with pole at the point corresponding to the center of S', are carried into circles concentric with S' and the trajectories orthogonal to these surfaces are carried into radii of S'.

In a second paper, to be published in a forthcoming issue of this Journal, we shall be interested in the correspondence brought about between the boundary points of D and those of D' when a correspondence of the above type is set up between the interior points. Our methods here are based essentially upon those employed by Osgood and Taylor in their proof of the fundamental theorem * that when a simply connected plane region S is mapped in a 1:1 and conformal manner upon the interior of the circle S', then an accessible point A, on the boundary of S, corresponds to a single point on the boundary of S', in the sense that the image of the curve along which A is accessible is a curve with a single limit point in the boundary of S'.

1.1 In regard to the quantity $| \nabla g(D \mid O, P) |$ the situation is somewhat different in three dimensions than it is in two. In fact, if S is a simply connected plane region and $g(S \mid O, P)$ its Green's function, then $| \nabla g(S \mid O, P) | > 0$ at every point P of S, independently of where in S the pole O is chosen. On the other hand, if T is a simply connected, three dimensional, normal domain, it is possible that $| \nabla g(T \mid O, P) |$ vanishes at some point P for some position of the pole O. This is shown to be the case in the following example.

Let T be the open set of points bounded by the torus t, obtained by revolving the circle

$$(y-4)^2 + z^2 = 1, \qquad x = 0,$$

about the z-axis. Let V_h be the open set of points determined by the inequalities

* Osgood and Taylor, "Conformal Transformations on the Boundaries of Their Regions of Definition," *Transactions of the American Mathematical Society*, Vol. 14 (1913), pp. 277-298.

$$x^2 + y^2 < 16, \qquad |z| < h \leq 1,$$

where $h > 0$, and let $S_h = T + U_h$.

The regions S_h and T are normal by several known conditions for normality, and S_h is simply connected. On the other hand, we can prove that $|\nabla g(S_h \mid A, P)|$ vanishes at some point P_h for small values of h, A being the point $(0, -4, 0)$.

This phenomenon is not surprising when the function $g(S_h \mid A, P)$ is interpreted as the potential at P due to a charge of electricity at A, the boundary of S_h being kept at zero potential. For by decreasing h we increase the resistance of the plate $S_h - T$ to the flow of electricity through it, while we do not change directly the flow through T. Thus for sufficiently small values of h one might expect that in the region T the potential function $g(S_h \mid A, P)$ behaves sensibly like $g(T \mid A, P)$. The latter of course, has an equilibrium point on the positive y-axis, and hence it is conceivable that so also has $g(S_h \mid A, P)$. That something like this actually is true we prove below.

We observe that the function

$$G(S_h \mid P) = g(S_h \mid A, P) - g(T \mid A, P)$$

is harmonic in T, if properly defined at A, and is continuous in $T + t$. Accordingly, because of the regularity of t,*

$$G(S_h \mid P) = (1/4\pi) \int_t \int G(S_h \mid Q) [\partial g(T \mid P, Q)/\partial n] \, d\sigma_Q,$$

where n is the interior normal to t, and Q is a point on the element of integration $d\sigma_Q$. This integral, of course, reduces to

$$(1) \qquad G(S_h \mid P) = (1/4\pi) \int_{\alpha(h)} \int g(S_h \mid A, Q) [\partial g(T \mid P, Q)/\partial n] d\sigma_Q,$$

where $\alpha(h)$ is the set of points on t determined by the inequalities

$$x^2 + y^2 \leq 16, \qquad |z| \leq h.$$

Now since $g(T \mid A, P)$ vanishes on t and is positive throughout T, we can find two points, P_1, P_2, on the segment of the positive y-axis interior to T, with $\overline{AP}_1 < \overline{AP}_2$, such that

$$g(T \mid A, P_1) = g(T \mid A, P_2) - \epsilon,$$

* Compare Goursat, *Cours d'Analyse*, Vol. 3 (1927), p. 523.

where ϵ is a sufficiently small positive quantity. At the point P_2 we have

$$g(S_h \mid A, P_2) > g(T \mid A, P_2),$$

since $G(S_h \mid P)$ is harmonic in T and non-negative on t, and at the point P_1 we have

$$g(S_h \mid A, P_1) - g(T \mid A, P_1) \leq J \text{ area } \alpha(h),$$

where J is a constant independent of h, since $\partial g(T \mid P_1, Q)/\partial n$ and $g(S_1 \mid A, Q)$ are continuous on t and the latter exceeds $g(S_h \mid A, Q)$ on $\alpha(h)$. Accordingly,

$$g(S_h \mid A, P_1) \leq J \text{ area } \alpha(h) - \epsilon + g(T \mid A, P_2).$$

Thus when h is small enough

$$g(S_h \mid A, P_1) < g(S_h \mid A, P_2). \tag{2}$$

From (2) and the fact that $\partial g(S_h \mid A, P)/\partial y < 0$ when P is on $\overline{AP}_1$, in the neighborhood of A, it follows that for some point P_h on $\overline{AP}_2$, $\partial g(S_h \mid A, P_h)/\partial y = 0$, when h is small enough. We can conclude from this the truth of the statement, for when P is on the y-axis we have, because of the position of the pole and the symmetry of S_h,

$$| \nabla g(S_h \mid A, P) | = | \partial g(S_h \mid A, P)/\partial y | .$$

2.1. *The Orthogonal Trajectories.* We proceed now to a discussion of the level surfaces, $g = \text{const.}$, and their orthogonal trajectories.

We choose the point O, the pole of $g(D \mid O, P)$ as the origin of a system of rectangular axes, x, y, z. The function g then takes the form

$$g(D \mid O, P) = g(P) = g(x, y, z) = 1/r + v(x, y, z),$$

where

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

and the function v is harmonic in D and continuous in the closed set $D + d$, d being the boundary of D.

Since g is harmonic and therefore analytic * in $D - O$, and since we

---

* The word "analytic" is used in the ordinary sense. A function $f(x_1, x_2, \cdots, x_n)$ in n variables is analytic in an open continuum S in the space of n dimensions if corresponding to each point $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ in S there is a number $\delta > 0$ such that throughout the domain $| x_1 - \bar{x}_1 |, | x_2 - \bar{x}_2 |, \cdots, | x_n - \bar{x}_n | \leq \delta$, $f(x_1, x_2, \cdots, x_n)$ is representable by a convergent power series in $x_1 - \bar{x}_1, x_2 - \bar{x}_2, \cdots, x_n - \bar{x}_n$. f is analytic in a closed continuum C if it is analytic in an open continuum S containing C.

Here and elsewhere in this paper by "convergence" we understand "absolute convergence." A detailed discussion of the properties of analytic functions of several real variables may be found in Riquier's *Les Systèmes d'Équations aux Dérivées Partielles*, Paris, 1910, Ch. 2, 3.

are supposing that ∇g does not vanish there, each level surface is a regular analytic surface, having a normal at each of its points.

Our interest in the trajectories orthogonal to these surfaces leads us to a consideration of the system of differential equations

$$\begin{aligned} dx/dt &= F_1(x,y,z) = g_x/(\nabla g)^2, \\ dy/dt &= F_2(x,y,z) = g_y/(\nabla g)^2, \\ dz/dt &= F_3(x,y,z) = g_z/(\nabla g)^2, \end{aligned} \tag{3}$$

where t is a parameter.

Now since g is analytic in $D-O$ and ∇g does not vanish there the functions F_i are likewise analytic there. We thus have at our disposal the classical existence theorem * on the solutions of (3). This may be stated as follows:

THEOREM A. *Let $P_0(x_0, y_0, z_0)$ denote any fixed point in $D-O$, t_0 an arbitrary real number, and F_{i,P_0} the element of F_i corresponding to the point P_0. There exists a positive number δ_{P_0} and a set of real functions $\phi_{i,P_0}(t, \xi, \eta, \zeta)$, $i=1,2,3$, of t and three independent real variables ξ, η, ζ such that*

(i) *$\phi_{i,P_0}(t, \xi, \eta, \zeta)$ is representable in a power series in $t-t_0$, $\xi-x_0$, $\eta-y_0$, $\zeta-z_0$ which converges for all $|t-t_0|, |\xi-x_0|, \cdots, |\zeta-z_0| < \delta_{P_0}$*

(ii) *$\phi_{1,P_0}(t_0, \xi, \eta, \zeta) = \xi$, $\phi_{2,P_0}(t_0, \xi, \eta, \zeta) = \eta$, $\phi_{3,P_0}(t_0, \xi, \eta, \zeta) = \zeta$;*

(iii) *$\partial\phi_{i,P_0}/\partial t = F_{i,P_0}(\phi_{1,P_0}, \phi_{2,P_0}, \phi_{3,P_0})$ $(i=1,2,3)$*

for all $|t-t_0|, |\xi-x_0|, \cdots, |\zeta-2z_0| < \delta_{P_0}$.
Moreover, the functions $\phi_{i,P_0}(t, x_0, y_0, z_0)$ of t form the only system of integrals of (3) which reduce to x_0, y_0, z_0 respectively when $t=t_0$.

This theorem is fundamental in our analysis.

A set of values of the variables ξ, η, ζ we may regard both as the coördinates of a point (ξ, η, ζ) in the (x, y, z)-space and likewise as defining together with t a point $p(t, \xi, \eta, \zeta)$ an Euclidean 4-space. The first interpretation is of value because the relations

$$x = \phi_{1,P_0}(t, \xi, \eta, \zeta), \quad y = \phi_{2,P_0}(t, \xi, \eta, \zeta), \quad z = \phi_{3,P_0}(t, \xi, \eta, \zeta),$$

then may be thought of as representing the trajectories as functions of their initial values. The second is of value because it enables us to employ the

* See for example, Goursat, *Cours d'Analyse*, Vol. 2 (1927), p. 182.

language and the results of the theory of analytic functions of several real variables. We shall have occasion to make use of both interpretations.

Our first theorem is concerned with the latter of these. More particularly it is concerned with the regions of existence in 4-space, and the properties therein, of the analytic functions which the various elements ϕ_{i,P_0} determine. These regions depend, of course, upon the initial point (x_0, y_0, z_0) and the initial value of the parameter t_0. Theorem I is concerned with those solutions for which $t_0 = 1$. It may be seen, however, by an inspection of the theorem that analogous results hold for solutions corresponding to an arbitrary value of t_0.

THEOREM I. *Let R represent the region in 4-space consisting of those points $p(t, \xi, \eta, \zeta)$ such that (ξ, η, ζ) is contained in $D - O$, and t exceeds $1 - g(\xi, \eta, \zeta)$. There exists one and only one set of functions $f_i(t, \xi, \eta, \zeta)$, $i = 1, 2, 3$, having the following properties in R:*

(a) *f_i is analytic in R;*

(b) *$f_1(1, \xi, \eta, \zeta) = \xi$, $f_2(1, \xi, \eta, \zeta) = \eta$, $f_3(1, \xi, \eta, \zeta) = \zeta$, for each point $p(1, \xi, \eta, \zeta)$ in R;*

(c) *the point $P(x, y, z)$, whose coördinates are $x = f_1(t, \xi, \eta, \zeta)$, $y = f_2(t, \xi, \eta, \zeta)$, $z = f_3(t, \xi, \eta, \zeta)$, is in $D - O$ and*

$$g(x, y, z) = t - 1 + g(\xi, \eta, \zeta), \qquad \partial f_i/\partial t = F_i(x, y, z) \qquad (i = 1, 2, 3), \tag{4}$$

for every p in R.

To prove this theorem we first make three preliminary observations relative to the functions ϕ_{i,P_0} referred to in Theorem A. Throughout this proof we shall suppose that the ϕ_{i,P_0} correspond to the initial value of the parameter $t_0 = 1$.

We observe firstly that if δ'_{P_0} be chosen less than * $g(P_0)/4$ and so small that the expansions of the ϕ_{i,P_0} converge for all $|t-1|, |\xi - x_0|, \cdots, |\zeta - z_0| < \delta'_{P_0}$, while δ''_{P_0} be chosen so as to satisfy this latter condition as well as to insure that (ξ, η, ζ) is in $D - O$ and $g(\xi, \eta, \zeta)$ exceeds $g(P_0)/2$ whenever $|\xi - x_0|, \cdots, |\zeta - z_0| < \delta''_{P_0}$, then the point $P(x, y, z)$, whose coördinates are

$$x = \phi_{1,P_0}(t, \xi, \eta, \zeta), \quad y = \phi_{2,P_0}(t, \xi, \eta, \zeta), \quad z = \phi_{3,P_0}(t, \xi, \eta, \zeta),$$

is in $D - O$ for all $|t-1| < \delta'_{P_0}$, $|\xi - x_0|, \cdots, |\zeta - z_0| < \delta''_{P_0}$.

* Since g is harmonic in $D - O$, positively infinite at O, and zero on d, it is positive at P_0.

For suppose the contrary to be true, that there be a set of values t', ξ', η', ζ' satisfying these conditions and such that the corresponding point P' is not in $D - O$. Then as t varies from one to t', P sweeps out an analytic arc. When $t = 1$, P coincides with (ξ', η', ζ'), a point in the δ''_{P_0}-neighborhood of P_0, and thus there is a number τ_1, on the open interval $(1, t')$,* such that when t is on the open interval $(1, \tau_1)$, P is contained in $D - O$, but when $t = \tau_1$, P lies on $d + O$. Hence, because g vanishes continuously on d and becomes positively infinite at O, we have

$$\lim_{t=\tau_1} g(x, y, z) = 0 \text{ or } \infty, \tag{5}$$

where the appropriate right-hand or left- hand limit is understood.

These relations enable us to reach a contradiction. In fact, we can choose a number τ_2 on the open interval $(1, \tau_1)$ such that for t on the open interval $(1, \tau_2)$, P lies in the δ''_{P_0}-neighborhood of P_0. Now since in this neighborhood

$$F_{i,P_0} = F_i \qquad (i = 1, 2, 3),$$

we must have

$$(\partial\phi_{i,P_0}/\partial t) = F_i(x, y, z) \qquad (i = 1, 2, 3),$$

for t in the open interval $(1, \tau_2)$. Thus, by definition of the F_i,

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\,\frac{\partial \phi_{1,P_0}}{\partial t} + \frac{\partial g}{\partial y}\,\frac{\partial \phi_{2,P_0}}{\partial t} + \frac{\partial g}{\partial z}\,\frac{\partial \phi_{3,P_0}}{\partial t} = 1,$$

or

$$g(x, y, z) = t - 1 + g(\xi, \eta, \zeta),$$

for these t. But g being analytic in $D - O$, this equation, because it holds for a small interval in t, must hold as long as P remains within $D - O$, thus for t on the open interval $(1, \tau_1)$. Accordingly,

$$\lim_{t=\tau_1} g(x, y, z) = g(\xi, \eta, \zeta) + \tau_1 - 1,$$

where the limit is taken as above. This quantity is bounded above and exceeds $g(P_0)/4$, which is greater than zero. This of course contradicts (5) and the desired result is established.

In the second place, we observe that the relations

$$\partial\phi_{i,P_0}/\partial t = F_i(\phi_{1,P_0}, \phi_{2,P_0}, \phi_{3,P_0}) \qquad (i = 1, 2, 3), \tag{6}$$

$$g(\phi_{1,P_0}, \phi_{2,P_0}, \phi_{3,P_0}) = t - 1 + g(\xi, \eta, \zeta) \tag{7}$$

hold for all $|t - 1| < \delta'_{P_0}$, $|\xi - x_0|, \cdots, |\zeta - z_0| < \delta''_{P_0}$, in as much as

* In designating this interval by $(1, t')$ we do not imply that $1 < t'$.

they hold in a small interval in t for each fixed set of values $|\xi - x_0|, \cdots, |\zeta - z_0| < \delta''_{P_0}$, and the point $(\phi_{1,P_0}, \phi_{2,P_0}, \phi_{3,P_0})$ remains within $D - O$ for all $|t-1| < \delta'_{P_0}$.

Finally, we note that corresponding to every closed subset U of $D - O$, there is a positive number δ_U such that the expansions of the functions ϕ_{i,P_0} in powers of $t-1$, $\xi - x_0, \cdots, \zeta - z_0$ converge for all $|t-1|, \cdots, |\zeta - z_0| < \delta_U$, independently of the position of P_0 in U. In fact, if this were not the case there would exist a sequence of points $\{P_n(x_n, y_n, z_n)\}$, all contained in U, and having a single limit point $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ in U, such that for each n the expansion of one of the functions of the set $\{\phi_{i,P_n}(t, \zeta, \eta, \xi)\}$, associated with P_n, would fail to converge for some $(t_n, \xi_n, \eta_n, \zeta_n)$ such that

$$|t_n - 1| = |\xi_n - x_n| = |\eta_n - y_n| = |z_n - \zeta_n| = 1/n.$$

Accordingly, we could choose a number n' so large that

$$|x_{n'} - \bar{x}|, \cdots, |z_{n'} - \bar{z}| < \delta''_{\bar{P}}/2 \text{ and } 1/n' < \delta''_{\bar{P}}/4,$$

where $\delta''_{\bar{P}}$ has the same meaning with regard to $\bar{P}$ as δ''_{P_0} has with regard to P_0. Now for every system of values $|\xi - x_{n'}|, \cdots, |\zeta - z_{n'}| < \delta''_{P_{n'}}$, each of the sets of functions $\{\phi_{i,\bar{P}}\}$, $\{\phi_{i,P_{n'}}\}$ of t, where the $\phi_{i,\bar{P}}$ are the functions associated with $\bar{P}$ by Theorem A, form a system of integrals of (3) and their members reduce to ξ, η, ζ respectively when $t = 1$. Hence, because of the uniqueness of such a set,

$$\phi_{i,\bar{P}} = \phi_{i,P_{n'}} \qquad (i = 1, 2, 3), \tag{8}$$

for all $|t-1|, \cdots, |\zeta - z'_n| < \delta''_{P_{n'}}$. From (8) and the fact that expansions of the $\phi_{i,\bar{P}}$ in powers of $t-1$, $\xi - x_{n'}, \cdots, \zeta - z_{n'}$, converge for all $|t-1|, \cdots, |\zeta - z_{n'}| < \delta''_{\bar{P}}/2$, it follows that the functions $\phi_{i,P_{n'}}$ have this latter property. We then reach a contradiction, for by hypothesis the expansions of the $\phi_{i,P_{n'}}$ failed to converge for

$$|t_{n'} - 1| = |\xi_{n'} - x_{n'}| = \cdots = |\zeta_{n'} - z_{n'}| = 1/n' < \delta'_{\bar{P}}/4.$$

With the aid of these preliminary remarks we are able to return to the main theorem and prove that with each point $\bar{p}(\bar{t}, \bar{\xi}, \bar{\eta}, \bar{\zeta})$ in R we can associate a neighborhood $C_{\bar{p}}$, defined by inequalities of the type $|t - t| < a$, $|\xi - \bar{\xi}|, \cdots, |\zeta - \bar{\zeta}| < b$, and a set of functions $f_{i\bar{p}}(t, \xi, \eta, \zeta)$, $\bar{\imath} = 1, 2, 3$, such that:

(d) $f_{i,\bar{p}}$ is representable in a power series in $t - \bar{t}, \cdots, \zeta - \bar{\zeta}$ which converges for all $p(t, \xi, \eta, \zeta)$ in $C_{\bar{p}}$;

(e) $f_{1,\bar{p}}(1, \xi, \eta, \zeta) = \xi$, $f_{2,\bar{p}}(1, \xi, \eta, \zeta) = \eta$, $f_{3,\bar{p}}(1, \xi, \eta, \zeta) = \zeta$, in case $\bar{p}$ is contained in the subset R' of R for which $t = 1$;

(f) The point $P(x, y, z)$, whose co-ordinates are

$$x = f_{1,\bar{p}}(t, \xi, \eta, \zeta), \; y = f_{2,\bar{p}}(t, \xi, \eta, \zeta), \; z = f_{3,\bar{p}}(t, \xi, \eta, \zeta),$$

is in $D - O$, and

$$g(x, y, z) = t - 1 + g(\xi, \eta, \zeta) \tag{9}$$
$$\partial f_{i,\bar{p}}/\partial t = F_i(x, y, z) \qquad (i = 1, 2, 3)$$

for all p in $C_{\bar{p}}$;

(g) On each segment $\xi = \text{const.}, \cdots, \zeta = \text{const.}$, contained in $C_{\bar{p}}$, the functions $f_{i,\bar{p}}$ are the continuations of functions $\alpha_i(t)$ which are analytic in the neighborhood of $t = 1$, reduce to ξ, η, ζ respectively when $t = 1$ and have the property that

$$d\alpha_i/dt = F_i(\alpha_1, \alpha_2, \alpha_3) \qquad (i = 1, 2, 3)$$

for t in this neighborhood.

If the truth of these assertions is once established the proof of the main theorem admits immediate conclusion. For, in the first place, the region R, being an open continuum, is one which permits definition of an analytic function therein. In the second place, it follows from (g) that if p_1, p_2 be any two points of R, C_{p_1}, C_{p_2} the neighborhoods, and f_{i,p_1}, f_{i,p_2} the functions associated with them, then

$$f_{i,p_1}(t, \xi, \eta, \zeta) = f_{i,p_2}(t, \xi, \eta, \zeta) \qquad (i = 1, 2, 3), \tag{10}$$

whenever p is in $C_{p_1} \cdot C_{p_2}$. In fact, for every p in $C_{p_1} \cdot C_{p_2}$ the functions $\alpha_i(t)$, corresponding to the three coördinates ξ, η, ζ of p, are uniquely determined and so also are their analytic extensions through real values of t. In the third place, because of (10) and (d), (e), (f) the functions $f_i(t, \xi, \eta, \zeta)$, defined at each point $\bar{p}$ in R as

$$f_i(\bar{t}, \bar{\xi}, \bar{\eta}, \bar{\zeta}) = f_{i\bar{p}}(\bar{t}, \bar{\xi}, \bar{\eta}, \bar{\zeta}) \qquad (i = 1, 2, 3),$$

have all the desired properties. Finally, the functions f_i are unique in as much as any set of functions having the properties (a), (b), (c) in R would, because of the uniqueness part of the theorem on the solutions of a differential system, have to agree with the f_i in the neighborhood of some point p of R', and thus, because of a well known property * of analytic functions, would have to agree with the f_i throughout R.

To obtain the functions $f_{i,\bar{p}}$ for a point $\bar{p}(1, \bar{\xi}, \bar{\eta}, \bar{\zeta})$ in R' we may apply

* Compare Riquier, *loc. cit.*, p. 86.

our preliminary results directly. In fact, $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ represents a point P_0 in R, and the functions

$$f_{i,\bar{p}}(t, \xi, \eta, \zeta) = \phi_{i,P_0}(t, \xi, \eta, \zeta) \qquad (i = 1, 2, 3)$$

have the desired properties (d), (e), (f) in the domain $C_{\bar{p}}$ defined by the inequalities

$$| t - 1 | < \delta'_{P_0}, | \xi - \bar{\xi} |, \cdots, | \zeta - \bar{\zeta} | < \delta''_{P_0}.$$

To obtain the functions $f_{i,\bar{v}}$ and establish their properties when $\bar{t} > 1$ one can proceed in much the same way as when $\bar{t} < 1$. We need consider, therefore, only the latter case.

We observe that when $\bar{p}$ is in R and $\bar{t} < 1$ the inequalities

$$g(\bar{\xi}, \bar{\eta}, \bar{\zeta}) > \bar{t} - 1 + g(\bar{\xi}, \bar{\eta}, \bar{\zeta}) = \epsilon' > 0$$

hold. Thus the set $U_{\bar{t}}$ in D determined by the inequalities

$$g(\bar{\xi}, \bar{\eta}, \bar{\zeta}) \geq g \geq \epsilon'$$

is well defined.

Now plainly $U_{\bar{t}}$ is closed and contains only points of $D - O$. Hence there exists a number $\delta_{U_{\bar{t}}} > 0$ so small that expansions in powers of $t - 1$, $\xi - x_0, \cdots, \zeta - z_0$ of the functions ϕ_{i,P_0}, associated with P_0 by Theorem A, converge for all $| t - 1 |, \cdots, | \zeta - z_0 | < \delta_{U_{\bar{t}}}$, independently of the location of $P_0(x_0, y_0, z_0)$ in $U_{\bar{t}}$.

Let ϵ denote the smaller of the two numbers $\epsilon'/4$, $\delta_{U_{\bar{t}}}$. Let p_0 denote the point whose coördinates are $1, \bar{\xi}, \bar{\eta}, \bar{\zeta}$, and let the segment running from p_0 to $\bar{p}$ be divided into n parts by the points of division

$$p_0(t_0, \bar{\xi}, \bar{\eta}, \bar{\zeta}) = p_0(1, \bar{\xi}, \bar{\eta}, \bar{\zeta}), p_1(t_1, \bar{\xi}, \bar{\eta}, \bar{\zeta}), \cdots, p_n(t_n, \bar{\xi}, \bar{\eta}, \bar{\zeta}) = \bar{p}(\bar{t}, \bar{\xi}, \bar{\eta}, \bar{\zeta}),$$

in such a way that $0 < t_j - t_{j+1} < \epsilon$, $j = 0, 1, \cdots, n - 1$.

We shall prove by induction that with each point p_j we can associate a set of functions $\{f_{i,p_j}(t, \xi, \eta, \zeta)\}$ and a positive number ϵ_j so small that in the neighborhood B_{p_j} determined by the inequalities $| t - t_j | < \epsilon$, $| \xi - \bar{\xi} |, \cdots$ $| \zeta - \bar{\zeta} | < \epsilon_j$, the functions f_{i,p_j} have the properties (d), (f), (g). When once this is done the proof of the theorem will be complete, for we may identify $C_{\bar{p}}$ with B_{p_n} and $f_{i,\bar{p}}$ with f_{i,p_n}.

We observe firstly that the statement is true for $j = 0$, with ϵ_0 chosen $\leq \delta_{U_{\bar{t}}}$ and so small that (ξ, η, ζ) is in $D - O$ and $g(\xi, \eta, \zeta) > g(\bar{\xi}, \bar{\eta}, \bar{\zeta})$, whenever $| \xi - \bar{\xi} |, \cdots, | \zeta - \bar{\zeta} | < \epsilon_0$. In fact, p_0 is a point R' and with it, as we have seen, there can be associated a neighborhood C_{p_0} and a set of functions having properties (d), (e), (f) in this domain. That these func-

tions have these properties throughout B_{p_0} is readily established from their very definition.

Suppose then that the statement is valid for the point p_j. Let

$$x_0 = f_{1,p}\ (t_{j+1}, \bar{\xi}, \bar{\eta}, \bar{\zeta}),\ y_0 = f_{2,p_j}(t_{j+1}, \bar{\xi}, \bar{\eta}, \bar{\zeta}),\ z_0 = f_{3,p_j}(t_{j+1}, \bar{\xi}, \bar{\eta}, \bar{\zeta}).$$

Then, because the f_{i,p_j} have property (f), we have

$$g(\bar{\xi}, \bar{\eta}, \bar{\zeta}) > g(x_0, y_0, z_0) = t_{j+1} - 1 + g(\bar{\xi}, \bar{\eta}, \bar{\zeta}) \geq \epsilon'.$$

Thus $P_0(x_0, y_0, z_0)$ is in $U\bar{\imath}$.

Consider then the functions $\phi_{i,P_0}(t, \xi, \eta, \zeta)$. From the theory of dominant functions it follows that since their expansion in powers of $t-1, \xi - x_0, \cdots, \zeta - z_0$ converge for all $|t-1|, |\xi - x_0|, \cdots, |\zeta - z_0| < \delta_{U\bar{\imath}}$, and since the expansions of the $f_{i,p_j}(t_{j+1}, \xi, \eta, \zeta)$ in powers of $\xi - \bar{\xi}, \eta - \bar{\eta}, \zeta - \bar{\zeta}$ converge for all $|\xi - \bar{\xi}|, \cdots, |\zeta - \bar{\zeta}| < \epsilon_j$ and reduce to x_0, y_0, z_0 respectively when $\xi - \bar{\xi}, \eta - \bar{\eta}, \zeta - \bar{\zeta}$, there is a positive number $\rho \leq \epsilon_j$ such that the expansions in powers of $t - t_{j+1}, \xi - \bar{\xi}, \cdots, \zeta - \bar{\zeta}$ of the functions

$$f_{i,p_{j+1}}(t, \xi, \eta, \zeta) = \phi_{i,P_0}[t - t_{j+1} + 1, f_{1,p_j}(t_{j+1}, \xi, \eta, \zeta), f_{2,p_j}, f_{3,p}] \quad (i = 1, 2, 3)$$

converge for all $|t - t_{j+1}| < \epsilon, |\xi - \bar{\xi}|, \cdots, |\zeta - \bar{\zeta}| < \rho$.

If then we choose $0 < \epsilon_{j+1} \leq \rho$ and so small that

$$(11) \quad |f_{1,p_j}(t_{j+1}, \xi, \eta, \zeta) - x_0|, \cdots, |f_{3,p_j}(t_{j+1}, \xi, \eta, \zeta) - z_0| < \delta''_{P_0},$$

when $|\xi - \bar{\xi}|, \cdots, |\zeta - \bar{\zeta}| < \epsilon_{j+1}$, it is readily verified that the functions $f_{i,p_{j+1}}$ have properties (d), (f), (g) in the neighborhood $B_{p_{j+1}}$. Property (d) is, in fact, a direct consequence of the preceding paragraph. Property (f), with the exception of (9), follows from (11) and the definitions of δ''_{P_0} and ϵ. As for (9) we have

$$\begin{aligned} g(f_{1,p_{j+1}}, \cdots, f_{3,p_{j+1}}) &= t - t_{j+1} + 1 - 1 + g[f_{1,p_j}(t_{j+1}, \xi, \eta, \zeta), \cdots, f_{3,p_j}] \\ &= t - t_{j+1} + t_{j+1} - 1 + g(\xi, \eta, \zeta) \\ &= t - 1 + g(\xi, \eta, \zeta). \end{aligned}$$

Finally, in regard to property (g) we observe that for any fixed set of numbers $|\xi - \bar{\xi}|, \cdots, |\zeta - \bar{\zeta}| < \epsilon_{j+1}$ both of the sets of functions $\{f_{i,p_j}\}, \{f_{i,p_{j+1}}\}$ of t are analytic in the neighborhood of $t = t_{j+1}$, reduce to $\{f_{i,p_j}(t_{j+1}, \xi, \eta, \zeta)\}$ (a point in $D - O$) when $t = t_{j+1}$, and form a system of integrals of (3). Since there is at most one such set it follows that

$$f_{i,p_j} = f_{i,p_{j+1}} \qquad (i = 1, 2, 3),$$

for $t_{j+1} + \epsilon > t > t_j - \epsilon$. Thus, since the f_{i,p_j} have property (g), so also have the $f_{i,p_{j+1}}$. Thus the induction is complete and Theorem I is proved.

2.2. In this section we shall give to the term 'orthogonal trajectory' a meaning which is precise and suited to the problem at hand.

We understand by an orthogonal trajectory one of those curves consisting of the points (x, y, z) determined by the system of formulas

$$x = \psi_1(\tau, x_0, y_0, z_0), \cdots, z = \psi_3(\tau, x_0, y_0, z_0), \tag{12}$$

when $P_0(x_0, y_0, z_0)$ is chosen on the surface $g = 1$ and then held fast, and τ is permitted to vary over the interval $0 \leq \tau < 1$, the functions ψ_i being defined as

$$\psi_i(\tau, x_0, y_0, z_0) = \begin{cases} f_i[(1-\tau)/\tau, x_0, y_0, z_0] & 0 < \tau < 1, \\ 0 & \tau = 0. \end{cases}$$

Conversely, every such curve is an orthogonal trajectory.

In regard to this definition two remarks should be made. In the first place, it should be noted that the functions ψ_i are well defined for $0 \leq \tau < 1$ and P_0 on $g = 1$, for the functions $f_i(t, x_0, y_0, z_0)$ are defined for $\infty > t > 0$ when P_0 is on $g = 1$. Moreover, since

$$g[f_1(t, x_0, \cdots, z_0), \cdots, f_3(t, \cdots, z_0)] = t,$$

when P_0 is on $g = 1$, and g is finite in $d + D - O$, it follows that

$$\lim_{\tau=0} f_i(t, x_0, \cdots, z_0) = 0,$$

or

$$\lim_{\tau=0} f_i[(1-\tau)/\tau, x_0, \cdots, z_0] = 0.$$

Consequently the functions ψ_i are continuous at $\tau = 0$, and the set $\{(x, y, z)\}$ of points (x, y, z) actually constitute a curve in the ordinary sense.

In the second place, it should be pointed out that this definition neither contradicts in any essential way the ordinary notions of an orthogonal trajectory, nor excludes from the family of trajectories any curve which might ordinarily be said to belong there. For on the one hand, every curve represented by (12) certainly cuts the level surfaces orthogonally, and on the other hand, it is readily verified, with the aid of the arguments in 2.52, that any curve, possessing at each of its points a tangent which coincides with the normal to the level surface through that point, coincides throughout its extent with one of the curves represented by (12).* The annexation of the point O to each trajectory is, of course, a slight extension of the ordinary

* Compare Morse, "Relations between the Critical Points of a Real Function of n Independent Variables," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 356-357. In this paper use is made of a differential system analogous to (3) for the study of a certain set of orthogonal trajectories.

concepts; our purpose in making this convention is simply to obtain a definite initial point for each trajectory.

2.3. By the preceding definition the trajectories are represented in terms of the point P_0 on the surface $g = 1$. In demonstrating the properties of the trajectories and in considering the mapping this representation is, however, by no means the most convenient one. We propose to prove in this section a theorem which shows that the trajectories, at least a certain group of them, can be represented in a very simple way in terms of the directions issuing from O. The proof of this theorem we base on the properties of the solution of the differential equations

$$d\sigma_i/d\mu = \Phi_i(\sigma_1, \sigma_2, \sigma_3) \qquad (i = 1, 2, 3)$$

obtained from the system

$$(13) \quad dx/d\mu = -|\nabla g| F_1,\ dy/d\mu = -|\nabla g| F_2,\ dz/d\mu = -|\nabla g| F_3$$

in changing over formally to the spherical coördinates σ_1, σ_2, σ_3, the colatitude σ_2 being measured from the positive z-axis and the longitude σ_3 from the positive x-axis, counterclockwise with regard to the positive z-axis. We have

$$(14) \quad \begin{aligned} x &= \theta_1(\sigma_1, \sigma_2, \sigma_3) = \sigma_1 \sin\sigma_2 \cos\sigma_3, \\ y &= \theta_2(\sigma_1, \sigma_2, \sigma_3) = \sigma_1 \sin\sigma_2 \sin\sigma_3, \\ z &= \theta_3(\sigma_1, \sigma_2, \sigma_3) = \sigma_1 \cos\sigma_2. \end{aligned}$$

The functions Φ_i have the form

$$\begin{aligned} \Phi_1(\sigma_1, \sigma_2, \sigma_3) &= (1 - \sigma_1^2 w_{\sigma_1})/K(\sigma_1, \sigma_2, \sigma_3), \\ \Phi_2(\sigma_1, \sigma_2, \sigma_3) &= -w_{\sigma_2}/K, \\ \Phi_3(\sigma_1, \sigma_2, \sigma_3) &= -w_{\sigma_3} \csc^2\sigma_2/K, \end{aligned}$$

where

$$(15) \quad w(\sigma_1, \sigma_2, \sigma_3) = v(\theta_1, \theta_2, \theta_3),$$

$$K(\sigma_1, \sigma_2, \sigma_3) = [(1 - \sigma_1^2 w_{\sigma_1})^2 + \sigma_1^2 w_{\sigma_2}^2 + \sigma_1^2 w_{\sigma_3}^2 \csc^2\sigma_3]^{1/2}.$$

The functions w, K, when σ_1, σ_2, σ_3 are regarded as the coördinates of a point in the Euclidean $(\sigma_1, \sigma_2, \sigma_3)$-space, are, of course, not necessarily analytic throughout the whole of this space, but certainly a positive number h can be chosen so small that in the region *

$$U_h: \quad |\sigma_1| < h,\ \pi/32 < \sigma_2 < 31\pi/32,\ \pi/32 < \sigma_3 < 63\pi/32,$$

these functions are defined through the formulas (14) and (15), and, in fact, are analytic there. Accordingly, the functions Φ_i are analytic there and by a slight extension of the theory on the solutions of a differential system,

* In particular, h is to be chosen so small that K is positive in this region.

there exists a set of functions $\lambda_i(\mu, \alpha, \beta)$, $i = 1, 2, 3$, of μ and two independent variables α, β and a number k_1 so small that in the region U_{k_1}

$$U_{k_1}: \quad |\mu| \leq k_1, \ \pi/16 < \alpha < 15\pi/16, \ \pi/16 < \beta < 31\pi/16,$$

in the (μ, α, β)-space, the functions λ_i have the following properties:

(i) λ_i is analytic in U_{k_1};

(ii) $\lambda_1(0, \alpha, \beta) = 0$, $\lambda_2(0, \alpha, \beta) = \alpha$, $\lambda_3(0, \alpha, \beta) = \beta$, for $(0, \alpha, \beta)$ in U_{k_1};

(iii) $(\lambda_1, \lambda_2, \lambda_3)$ represents a point in U_h and

$$\partial\lambda_i/\partial\mu = \Phi_i(\lambda_1, \lambda_2, \lambda_3) \qquad (i = 1, 2, 3)$$

for (μ, α, β) in U_{k_1}.

In employing these facts it is not difficult to prove

THEOREM II. *Let* Γ, Γ_1 *represent the sets of points* $q(\tau, \alpha, \beta)$ *in the* (τ, α, β)*-space defined by the inequalities*

Γ: $0 \leq \tau < 1, \ \pi/8 < \alpha < 7\pi/8, \ \pi/8 < \beta < 15\pi/8,$

Γ_1: $0 < \tau < 1, \ \pi/8 < \alpha < 7\pi/8, \ \pi/8 < \beta < 15\pi/8.$

There exists a set of functions $\phi_i(\tau, \alpha, \beta)$, $i = 1, 2, 3$, *having the following properties in* Γ, Γ_1:

(a) $\phi_i(\tau, \alpha, \beta)$ *is analytic in* Γ_1, *and continuous in* Γ *with* $\phi_i(0, \alpha, \beta) = 0$;

(b) *the point* $P(x, y, z)$, *whose coördinates are*

$$x = \phi_1(\tau, \alpha, \beta), \ y = \phi_2(\tau, \alpha, \beta), \ z = \phi_3(\tau, \alpha, \beta),$$

is in $D - O$, *and*

$$g(x, y, z) = 1/\tau - 1$$
$$\partial\phi_i/\partial\tau = -F_i(x, y, z)/\tau^2 \qquad (i = 1, 2, 3),$$

for q *in* Γ_1;

(c) *the right-hand derivative* $(\partial^+\phi_i/\partial\tau^+)_{\tau=0, \ \alpha=\bar{\alpha}, \ \beta=\bar{\beta}}$, *where*

$$\pi/8 < \alpha < 7\pi/8, \ \pi/8 < \beta < 15\pi/8,$$

exists and is equal to

$$\sin\bar{\alpha}\cos\bar{\beta}, \ \sin\bar{\alpha}\sin\bar{\beta}, \ \cos\bar{\alpha},$$

depending upon whether $i = 1$, 2 *or* 3;

(d) *the Jacobian*

$$\partial(\phi_1, \phi_2, \phi_3)/\partial(\tau, \alpha, \beta) = \sin\alpha/(\tau^2\nabla g)^2 \tag{16}$$

for every q *in* Γ_1.

It is to be observed that this theorem shows that if ρ be any ray issuing from O which makes an angle $\pi/8 < \alpha < 7\pi/8$ with the positive z-axis and whose projection on the (x, y)-plane makes an angle $\pi/8 < \beta < 15\pi/8$ with the positive x-axis, β being measured counterclockwise with regard to the positive z-axis, then there is a trajectory $C_{\alpha,\beta}$

$$x = \phi_1(\tau, \alpha, \beta),\ y = \phi_2(\tau, \alpha, \beta),\ z = \phi_3(\tau, \alpha, \beta) \tag{17}$$

which has ρ as its forward tangent at O. In fact, by (b) $C_{\alpha,\beta}$ in an orthogonal trajectory, and by (c) it has ρ as its forward tangent at O.

The proof of the theorem we divide into three parts. In (A) the functions ϕ_i are defined and shown to have properties (a), (b). In (B) and (C) these functions are shown to have properties (c) and (d).

(A) We first pass from the functions $\lambda_i(\mu, \alpha, \beta)$ noted above to functions $\bar{\lambda}_i$ of τ, α, β in the following way. We observe that $w(\lambda_1, \lambda_2, \lambda_3)$ is bounded and that $K(\lambda_1, \lambda_2, \lambda_3) > 0$ for (μ, α, β) in U_{k_1}. Thus, since

$$\partial\lambda_1/\partial\mu = [1 - \lambda_1^2 w_{\sigma_1}(\lambda_1, \lambda_2, \lambda_3)]/K(\lambda_1, \lambda_2, \lambda_3),$$

there is a number $0 < k_2 < k_1$ so small that for each (μ, α, β) in the subset

$$U_{k_2}:\ 0 < \mu \leq k_2,\ \pi/8 \leq \alpha \leq 7\pi/8,\ \pi/8 \leq \beta \leq 15\pi/8$$

of U_{k_1}, the function $\partial\lambda_1/\partial\mu > 0$. Thus, for each pair of values

$$\pi/8 \leq \alpha \leq 7\pi/8,\ \pi/8 \leq \beta \leq 15\pi/8, \tag{18}$$

the function λ_1 of μ increases strictly as μ increases from zero to k_2. Accordingly, the function

$$H(\mu, \alpha, \beta) = 1/\lambda_1 + w(\lambda_1, \lambda_2, \lambda_3)$$

is analytic in U_{k_2}. Moreover, since

$$\frac{\partial H}{\partial \mu} = -\frac{1}{\lambda_1^2}\ \frac{\partial \lambda_1}{\partial \mu} + \sum_{i=1}^{3} w_{\sigma_i}\ \frac{\partial \lambda_i}{\partial \mu} = -\frac{K(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1^2}, \tag{19}$$

it follows that, for each fixed pair of values α, β satisfying (18), H increases strictly as μ decreases from k_2 to zero.

Let then $1/\bar{h} - 1$ denote the upper bound of $H(k_2, \alpha, \beta)$ for (α, β) in (18). Then, for every $0 < \tau \leq \bar{h}$ and (α, β) in (18), there is one and only one value $0 < \mu \leq k_2$ for which the equation

$$H(\mu, \alpha, \beta) = 1/\tau - 1$$

is satisfied. This equation thus defines μ as a single valued function

$$\mu = \lambda(\tau, \alpha, \beta)$$

of the variables τ, α, β. That this function is analytic in the region

$$U_{\bar{h}}: \quad 0 < \tau < \bar{h},\ \pi/8 < \alpha < 7\pi/8,\ \pi/8 < \beta < 15\pi/8,$$

in the (τ, α, β)-space, is an immediate consequence of the fact that H is analytic and $\partial H/\partial\mu \neq 0$ for (μ, α, β) in U_{k_2}. We have, moreover, (λ, α, β) representing a point in U_{k_2} and

$$H(\lambda, \alpha, \beta) = 1/\lambda_1(\lambda, \alpha, \beta) + w(\lambda_1, \lambda_2, \lambda_3) = 1/\tau - 1 \tag{20}$$

when $q(\tau, \alpha, \beta)$ is in $U_{\bar{h}}$, and

$$\lim_{\tau=0} \lambda(\tau, \alpha, \beta) = 0 \tag{21}$$

for each $\pi/8 < \alpha < 7\pi/8$, $\pi/8 < \beta < 15\pi/8$.

The functions $\bar{\lambda}_i$ are then by definition

$$\bar{\lambda}_i(\tau, \alpha, \beta) = \lambda_i(\lambda, \alpha, \beta) \qquad (i = 1, 2, 3).$$

They have the properties:

(e) $\bar{\lambda}_i$ is analytic in $U_{\bar{h}}$;

(f) $\lim_{\tau=0} \bar{\lambda}_1 = 0$, $\lim_{\tau=0} \bar{\lambda}_2 = \alpha$, $\lim_{\tau=0} \bar{\lambda}_3 = \beta$, for each $\pi/8 < \alpha < 7\pi/8$, $\pi/8 < \beta < 15\pi/8$;

(g) $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$ is in U_h and

$$\frac{\partial \bar{\lambda}_i}{\partial \tau} = \frac{1}{\tau^2} \frac{\partial \lambda_i}{\partial \mu} \frac{\bar{\lambda}_1^2}{K(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)} = \frac{\bar{\lambda}_1^2 \Phi_i(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)}{\tau^2 K(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)}$$

for each q in $U_{\bar{h}}$.

Property (e) is a consequence of the analyticity of the λ_i, λ. Property (f) follows from (ii) and (21). Property (g) follows from (iii), (19), (20).

We now pass from the $\bar{\lambda}_i$ to the ϕ_i of the theorem. We let

$$\begin{aligned} \bar{\phi}_1(\tau, \alpha, \beta) &= \bar{\lambda}_1 \sin \bar{\lambda}_2 \cos \bar{\lambda}_3, \\ \bar{\phi}_2(\tau, \alpha, \beta) &= \bar{\lambda}_1 \sin \bar{\lambda}_2 \sin \bar{\lambda}_3, \\ \bar{\phi}_3(\tau, \alpha, \beta) &= \bar{\lambda}_1 \cos \bar{\lambda}_2, \end{aligned} \tag{22}$$

and observe that

(h) $\bar{\phi}_i$ is analytic in $U_{\bar{h}}$;

(j) $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ is a point in D and

$$\begin{aligned} g(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) &= 1/\bar{\lambda}_1 + v(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \\ &= 1/\bar{\lambda}_1 + w(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) \\ &= 1/\tau - 1 \end{aligned}$$

when q is in $U_{\bar{h}}$, the first of these equations being a consequence of (22) and the fact that $\bar{\lambda}_1 > 0$ when q is in $U_{\bar{h}}$, the second following from (16), and the last from (20).

We then define

$$\phi_i(\tau, \alpha, \beta) = f_i(t, \xi, \eta, \zeta) \qquad (i = 1, 2, 3), \tag{23}$$

where

$$t = 1/\tau - 2/\bar{h} + 1,$$

$$\xi = \bar{\phi}_1(\bar{h}/2, \alpha, \beta),\ \eta = \bar{\phi}_2(\bar{h}/2, \alpha, \beta),\ \zeta = \bar{\phi}_3(\bar{h}/2, \alpha, \beta),$$

for q in Γ_1, and

$$\phi_i(0, \alpha, \beta) = 0 \qquad (i = 1, 2, 3)$$

for $q(0, \alpha, \beta)$ in Γ.

It is readily verified that the ϕ_i have properties (a), (b). In the first place, the definition is permissible and the functions ϕ_i are analytic in Γ_1 because of (h), Theorem I, and the fact that

$$t = 1/\tau - 2/\bar{h} + 1 > 2 - 2/\bar{h} = 1 - g(\xi, \eta, \zeta).$$

In the second place, by Theorem I, (j) and (23), (ϕ_1, ϕ_2, ϕ_3) represents a point in $D - O$, and

$$\begin{aligned} g(\phi_1, \phi_2, \phi_3) &= g(f_1, f_2, f_3) \\ &= t - 1 + g(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \\ &= 1/\tau - 2/\bar{h} + 1 + 2/\bar{h} - 1 \\ &= 1/\tau - 1. \end{aligned}$$

In the third place,

$$(\phi_1^2 + \phi_2^2 + \phi_3^2)^{-1/2} = 1/\tau - 1 - v > 1/\tau - 1$$

so that

$$(\phi_1^2 + \phi_2^2 + \phi_3^2)^{1/2} < \tau/(1 - \tau)$$

and the functions are continuous in Γ. Finally, in the fifth place, by Theorem I and (23)

$$\frac{\partial \phi_i}{\partial \tau} = \frac{\partial f_i}{\partial t}\,\frac{\partial t}{\partial \tau} = -\frac{1}{\tau^2} F_i(\phi_1, \phi_2, \phi_3) \qquad (i = 1, 2, 3).$$

(B) The ϕ_i thus have properties (a), (b). In order to show that they likewise have property (c) we prove that the $\bar{\phi}_i$ have this property and that

$$\bar{\phi}_i(\tau, \alpha, \beta) = \phi_i(\tau, \alpha, \beta) \qquad (i = 1, 2, 3) \tag{24}$$

when q is in $U_{\bar{h}}$.

Let

$$\bar{\phi}_i(0, \alpha, \beta) = 0.$$

Then in virtue of (j) and (22)

$$\bar{\phi}_1/\tau = \sin \bar{\lambda}_2 \cos \bar{\lambda}_3 + \bar{\phi}_1[v(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) + 1].$$

Thus since v is bounded in $D + d$ and

$$0 < \bar{\phi}_1 < \tau/(1 - \tau)$$

when q is in $U_{\bar{h}}$, it follows by (j) that

$$\lim_{\tau=0} [\bar{\phi}_1(\tau, \alpha, \beta)/\tau] = \sin\alpha \cos\beta.$$

That is,

$$(\partial^+\bar{\phi}_1/\partial\tau^+)_{t=0,\ \alpha=a,\ \beta=\beta} = \sin\alpha\cos\beta.$$

Similarly,

$$(\partial^+\bar{\phi}_2/\partial\tau^+)_{t=0} = \sin\alpha\sin\beta, \quad (\partial^+\bar{\phi}_3/\partial\tau^+)_{t=0} = \cos\alpha.$$

We have then only to verify (24). For this we observe that

$$\partial\bar{\phi}_1/\partial\tau = (\partial\bar{\lambda}_1/\partial\tau)\sin\bar{\lambda}_2\cos\bar{\lambda}_3 + \bar{\lambda}_1\cos\bar{\lambda}_2\cos\bar{\lambda}_3\,(\partial\bar{\lambda}_2/\partial\tau) - \bar{\lambda}_1\sin\bar{\lambda}_2\sin\bar{\lambda}_3\,(\partial\bar{\lambda}_3/\partial\tau),$$

which by (g) becomes

$$= \bar{\lambda}_1^2(\sin\bar{\lambda}_2\cos\bar{\lambda}_3\Phi_1 + \bar{\lambda}_1\cos\bar{\lambda}_2\cos\bar{\lambda}_3\Phi_2 - \bar{\lambda}_1\sin\bar{\lambda}_2\sin\bar{\lambda}_3\Phi_3)/(\tau^2K^2).$$

Thus, in making use of the formula

$$w_{\sigma_1}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) = v_x(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)\sin\bar{\lambda}_2\cos\bar{\lambda}_3 + v_y\bar{\lambda}_1\cos\bar{\lambda}_2\cos\bar{\lambda}_3 - v_z\bar{\lambda}_1\sin\bar{\lambda}_2$$

and the other two of similar nature, we find

$$\partial\bar{\phi}_1/\partial\tau = -F_1(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)/\tau^2. \tag{25}$$

In the same manner,

$$\partial\bar{\phi}_i/\partial\tau = -F_i(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)/\tau^2 \qquad (i = 2, 3).$$

Now for $\tau = \bar{h}/2$, we have

$$\phi_i(\bar{h}/2, \alpha, \beta) = \bar{\phi}_i(\bar{h}/2, \alpha, \beta) \qquad (i = 1, 2, 3).$$

This, together with (a), (b), (h), (j) and the uniqueness theorem on the integrals of a different system, is sufficient to establish (24).

(C) To prove that

$$\partial(\phi_1, \phi_2, \phi_3)/\partial(\tau, \alpha, \beta) = \sin\alpha/\tau^2(\nabla g)^2,$$

we first compute $\partial J/\partial\tau$ where,

$$J(\tau, \alpha, \beta) = \tau^2(\nabla g)^2\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(\tau, \alpha, \beta)} = -\begin{vmatrix} g_x & \frac{\partial\phi_1}{\partial\alpha} & \frac{\partial\phi_1}{\partial\beta} \\ g_y & \cdot\cdot & \cdot\cdot \\ g_z & \cdot\cdot & \frac{\partial\phi_3}{\partial\beta} \end{vmatrix}.$$

We find upon making use of (b) and the fact that g is harmonic in $D - O$, that

$$\partial J/\partial\tau = 0.$$

Accordingly, J is a function of α, β alone.

Now

$$J = \begin{vmatrix} \phi_1/\tau - \phi_1 - v\phi_1 - v_x r^2 & (\partial\phi_1/\partial\alpha)/r & (\partial\phi_1/\partial\beta)/r \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \phi_3/\tau - \phi_3 - v\phi_3 - v_x r^2 & \cdot & (\partial\phi_3/\partial\beta)/r \end{vmatrix},$$

where $r = (\phi_1^2 + \phi_2^2 + \phi_3^2)^{\frac{1}{2}}$, and we know that for each fixed set of values

$$\pi/8 < \alpha < 7\pi/8,\ \pi/8 < \beta < 15\pi/8, \tag{26}$$

we have

$$\lim_{\tau=0}[\phi_1/\tau] = \sin\alpha\cos\beta,\ \lim_{\tau=0}[\phi_2/\tau] = \sin\alpha\sin\beta,\ \lim_{\tau=0}[\phi_3/\tau] = \cos\alpha,$$

$$\lim_{\tau=0}\phi_i = \lim_{\tau=0}[v\phi_i] = \lim_{\tau=0}[r^2(\nabla v)^2] = 0.$$

Consequently, if we can show that for each fixed pair (α, β) in (26)

$$\lim_{\tau=0}[(\partial\phi_i/\partial\alpha)/r] = \begin{cases} \cos\alpha\cos\beta \\ \cos\alpha\sin\beta, \\ -\sin\alpha \end{cases} \quad \lim_{\tau=0}[(\partial\phi_i/\partial\beta)/r] = \begin{cases} -\sin\alpha\sin\beta & i=1, \\ \sin\alpha\cos\beta & i=2, \\ 0 & i=3, \end{cases} \tag{27}$$

the theorem will be proved, for if these numbers are substituted in their proper places in the above determinant the resulting value is $\sin\alpha$.

All of the limits (27) can be verified by a process similar to that we use for the first.

We have by (22), for an arbitrary pair of values α, β in (26),

$$H(\lambda, \alpha, \beta) = 1/\lambda_1(\lambda, \alpha, \beta) + w(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) = 1/\tau - 1.$$

Consequently, for all $0 < \tau < \bar{h}$,

$$\partial\lambda/\partial\alpha[(\partial\lambda_1/\partial\mu) - \lambda_1^2 \sum_{i=1}^{3} w_{\sigma_i}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)(\partial\lambda_i/\partial\mu)] = -(\partial\lambda_1/\partial\alpha) + \bar{\lambda}_1^2 \sum_{i=1}^{3} w_{\sigma_i}(\partial\lambda_i/\partial\alpha). \tag{28}$$

Now $\partial\lambda_i/\partial\mu$, $\partial\lambda_i/\partial\alpha$ are bounded for all $0 < \tau < \bar{h}$, and

$$\lim_{\tau=0}\lambda_i = 0,\ \lim_{\tau=0}(\partial\lambda_1/\partial\alpha) = 0,\ \lim_{\tau=0}(\partial\lambda_1/\partial\mu) = 1, \tag{29}$$

the latter two of these limits resulting from (i), (ii) and the definition of the Φ_i. Thus it follows from (28) that

$$\lim_{\tau=0}(\partial\lambda/\partial\alpha) = 0. \tag{30}$$

The relation (28) may, on the other hand, be written as

$$\frac{1}{\bar{\lambda}_1}\left(\frac{\partial\lambda_1}{\partial\mu}\ \frac{\partial\lambda}{\partial\alpha} + \frac{\partial\lambda_1}{\partial\alpha}\right) = \bar{\lambda}_1 \sum_{i=1}^{3} w_{\sigma_i}\left(\frac{\partial\lambda_i}{\partial\mu}\ \frac{\partial\lambda}{\partial\alpha} + \frac{\partial\lambda_i}{\partial\alpha}\right)$$

In passing to the limit here we get as a result of (29) and (30)

$$\lim_{\tau=0}\left[\left(\frac{\partial\lambda_1}{\partial\mu}\ \frac{\partial\lambda}{\partial\alpha}+\frac{\partial\lambda_1}{\partial\alpha}\right)/\bar{\lambda}_1\right]=0. \tag{31}$$

This relation enables us to establish the first of the limits in (27). In fact, by (22)

$$\begin{aligned}\frac{1}{r}\ \frac{\partial\phi_1}{\partial\alpha}=\frac{1}{r}\ \frac{\partial\bar{\phi}_1}{\partial\alpha}&=\frac{1}{\lambda_1}\left(\frac{\partial\lambda_1}{\partial\mu}\ \frac{\partial\lambda}{\partial\alpha}+\frac{\partial\lambda_1}{\partial\alpha}\right)\sin\bar{\lambda}_2\cos\bar{\lambda}_3\\ &+\left(\frac{\partial\lambda_2}{\partial\mu}\ \frac{\partial\lambda}{\partial\alpha}+\frac{\partial\lambda_2}{\partial\alpha}\right)\cos\bar{\lambda}_2\cos\bar{\lambda}_3\\ &-\left(\frac{\partial\lambda_3}{\partial\mu}\ \frac{\partial\lambda}{\partial\alpha}+\frac{\partial\lambda_3}{\partial\alpha}\right)\sin\bar{\lambda}_2\sin\bar{\lambda}_3.\end{aligned} \tag{32}$$

But by (ii)

$$\lim_{\tau=0}\ (\partial\lambda_2/\partial\alpha)=1,\quad \lim_{\tau=0}\ (\partial\lambda_3/\partial\alpha)=0.$$

Thus by (31), (32)

$$\lim_{\tau=0}\left(\frac{1}{r}\ \frac{\partial\phi_1}{\partial\alpha}\right)=\cos\alpha\cos\beta,$$

and the theorem is proved.

2.4. The theorem of the preceding section gives us, as we have seen, a representation of a certain group of trajectories, the group being restricted by the bounds upon the variables α, β and thus upon the directions issuing from O. This group may be enlarged and all the directions issuing from O accounted for by a theorem of the same nature as Theorem II. This new theorem will show that if ρ be any ray issuing from O which makes an angle $\pi/8<\alpha'<7\pi/8$, with the positive y-axis and whose projection on the (z,x)-plane makes an angle β' with the negative x-axis, β' being measured counterclockwise with regard to the positive y-axis, then there is a trajectory

$$x=\phi_1'(\tau,\alpha,\beta),\quad y=\phi_2'(\tau,\alpha',\beta'),\quad z=\phi_3'(\tau,\alpha',\beta')$$

having ρ as its forward tangent at O.

The proof of this theorem is based upon the properties of the solutions of the system of differential equations

$$d\sigma_i/d\mu=\Phi_i'(\sigma_1,\sigma_2,\sigma_3)\qquad(i=1,2,3)$$

obtained from the system (13) in changing over to the spherical coördinates $\sigma_1,\sigma_2,\sigma_3$, the colatitude σ_2 being measured in this case from the positive y-axis, and the longitude σ_3 from the negative x-axis, counterclockwise with regard to the positive y-axis. Since, however, the details differ in no essential way from those of Theorem II we shall omit the proof.

THEOREM II'. *Let* Γ', Γ_1' *represent the sets of points* $q'(\tau,\alpha',\beta')$ *in the* (τ,α',β') *space defined by the inequalities*

$$\Gamma': \quad 0 \leq \tau < 1, \ \pi/8 < \alpha' < 7\pi/8, \ \pi/8 < \beta' < 15\pi/8,$$
$$\Gamma_1': \quad 0 < \tau < 1, \ \pi/8 < \alpha' < 7\pi/8, \ \pi/8 < \beta' < 15\pi/8.$$

There exists a set of functions $\phi_i'(\tau, \alpha', \beta')$, $i = 1, 2, 3$, *having the following properties in* Γ_1, Γ_1':

(a) $\phi_i'(\tau, \alpha', \beta')$ *is analytic in* Γ_1' *and continuous in* Γ' with $\phi_i(0, \alpha', \beta') = 0$;

(b) *the point* $P(x, y, z)$, *whose co-ordinates are*

$$x = \phi_1'(\tau, \alpha', \beta'), \quad y = \phi_2'(\tau, \alpha', \beta'), \quad z = \phi_3'(\tau, \alpha', \beta'),$$

is in $D - O$, *and*

$$g(x, y, z) = 1/\tau - 1,$$
$$\partial\phi_i'/\partial\tau = -F_i(x, y, z)/\tau^2 \qquad (i = 1, 2, 3)$$

for q' *in* Γ_1';

(c) *the right-hand derivative* $(\partial^+\phi_i'/\partial\tau^+)_{t=0,\ \alpha'=\bar{\alpha}',\ \beta'=\bar{\beta}'}$, *where*

(33) $$\pi/8 < \bar{\alpha}' < 7\pi/8, \quad \pi/8 < \bar{\beta}' < 15\pi/8$$

exists and is equal to

$$-\sin\bar{\alpha}'\cos\bar{\beta}', \quad \cos\bar{\alpha}', \quad \sin\bar{\alpha}'\sin\bar{\beta}',$$

depending upon whether $i = 1$, 2, or 3;

(d) *the Jacobian*

$$[\partial(\phi_1', \phi_2', \phi_3')/\partial(\tau, \alpha, \beta)] = \sin\alpha'/[\tau^2(\nabla g)^2]$$

for every q' *in* Γ_1'.

It is evident from (b) and (c) that the various curves $C'_{\alpha',\beta'}$

$$x = \phi_1'(\tau, \alpha', \beta'), \quad y = \phi_2'(\tau, \alpha'\beta'), \quad z = \phi_3'(\tau, \alpha', \beta')$$

are orthogonal trajectories and have the correct forward tangents at O.

Whether the two families of curves $\{C_{\alpha,\beta}\}$, $\{C'_{\alpha'',\beta''}\}$ include all the trajectories we cannot as yet say. For there may be rays issuing from O which are the forward tangents of several trajectories, the families $\{C_{\alpha,\beta}\}$, $\{C'_{\alpha',\beta'}\}$ including at most two of these, while again there may be trajectories having no tangents at O. We shall see in the next section, however, that the two families do comprise all the trajectories. This next section, 2. 5, is divided into several subsections 2. 51, 2. 52, $\cdots$, 2. 55, in each of which a property of the trajectories is given.

2. 51. *Each trajectory* T_{P_0} *has at least one limit point on* d, *and all the limit points of* T_{P_0} *not contained in* T_{P_0} *are points of* d.

To prove the first part of this statement we observe that the points $P^{(n)}$ on T_{P_0} corresponding to the values $\tau = 1 - 1/n$, where n is an integer exceeding 2, have a limit point $\bar{P}$ in $D + d$. $\bar{P}$ cannot, however, lie in D for g is positive and continuous in $D - O$, and becomes infinite continuously at O, while

$$\lim_{n=\infty} g(P_n) = \lim_{n=\infty} [1/(n-1)] = 0.$$

The truth of the second part follows from the facts that T_{P_0} is an analytic arc and contains O, that g is continuous in $D - O$, and finally that

$$g(\psi_1, \psi_2, \psi_3) = (1-\tau)/\tau$$

2.52. *Through every point of $D - O$ there passes one and only one orthogonal trajectory.*

To prove this let us denote by $I_{P'}$ the curve determined by the formulas

$$x = f_1(t, x', y', z'), \; y = f_2(t, \cdots, z'), \; z = f_3(t, \cdots, z')$$

when $P'(x', y', z')$ is fixed in $D - O$, and t is permitted to vary over the interval $\infty > t > 1 - g(x', y', z')$.

Then through each point of $D - O$ there passes an $I_{P'}$, since (x, y, z) coincides with (x', y', z') when $t = 1$. Moreover, in as much as

$$g(x, y, z) = t - 1 + g(x', y', z'),$$

the curve $I_{P'}$ intersects the surface $g = \alpha > 0$, once and only once, namely, when

$$t = \alpha + 1 - g(x', y', z').$$

Further, if $P'(x', y', z')$, $P''(x'', y'', z'')$ be any two points of $D - O$, and $I_{P'}$, $I_{P'}$ their corresponding curves, then $I_{P'}$, $I_{P'}$ coincide throughout if they have a single point $(\bar{x}, \bar{y}, \bar{z})$ in common. In fact, the curve $I_{P'}$ may be defined by the formulas

$$x = f_1[t + 1 - g(x', y', z'), x', y', z'], \cdots, z = f_3(t +, \cdots, z'), \quad \infty > t > 0,$$

and the curve $I_{P''}$ admits similar representation with x', y', z' replaced by x'', y'', z'', respectively. We have in both cases

$$x = \bar{x}, \quad y = \bar{y}, \quad z = \bar{z},$$

when $t = g(\bar{x}, \bar{y}, \bar{z}) = \bar{t}$. Thus, as a consequence of the fact that there is at most one system of integrals of (3), analytic in t in the neighborhood of $t = \bar{t}$, which reduce to $\bar{x}, \bar{y}, \bar{z}$ when $t = \bar{t}$, it follows that

$$f_i[t + 1 - g(x', y', z'), x', y', z'] = f_i[t + 1 - g(x'', y'', z''), x'', y'', z''] \quad (i = 1, 2, 3)$$

in the neighborhood of $\bar{t}$, hence throughout $\infty > t > 0$. Accordingly, $I_{P'}$ is identical with $I_{P''}$.

From these facts it follows that if P' be any point in $D - O$, there is one and only one trajectory through P'. In fact, the trajectory T_{P_0}, where P_0 is the point of intersection of $I_{P'}$ with $g = 1$, has the desired property, since it contains all the points of I_{P_0} and consequently of $I_{P'}$. On the other hand, no trajectory other than T_{P_0} can contain P'. For if l be any trajectory distinct from T_{P_0} it contains a point P_1, on $g = 1$ distinct from P_0. Now l, except for the point O, is identical with I_{P_1}, but I_{P_1} does not contain P_0 and therefore cannot contain P'. Consequently l cannot contain P'.

2. 53. *Each trajectory is rectifiable over the interval* $0 \leq \tau \leq \frac{1}{2}$.

To prove this we show that the functions $\psi_i(\tau, x_0, y_0, z_0)$ are of bounded variation in this interval for each P_0 on $g = 1$. We have, for any partition of the interval $0 = \tau_1 < \tau_2 \cdots < \tau_n = \frac{1}{2}$

$$\sum_{j=1}^{n-1} | \psi_i(\tau_{j+1}, x_0, y_0, z_0) - \psi_i(\tau_j, x_0, y_0, z_0) | \leq | \psi_i(\tau_1, x_0, y_0, z_0) | + \int_{\tau_1}^{\frac{1}{2}} | \partial\psi_i/\partial\tau | \, d\tau$$

$$\leq | \psi_i(\tau_1, x_0, y_0, z_0) | + \int_{\tau_1}^{\frac{1}{2}} [1/(\tau^2 | \nabla g |)] dt,$$

by the definition of ψ_i.

Now

$$(\nabla g)^2 = [1 + 2r^2(\nabla r) \cdot (\nabla v) + r^4(\nabla v)^2]/r^4,$$

and

$$g(\psi_1, \psi_2, \psi_3) = 1/r + v = 1/\tau - 1,$$

so that

$$\tau^4(\nabla g)^2 = [1 + 2r^2(\nabla r) \cdot (\nabla v) + r^4(\nabla v)^2]/(1 - r + rv)^4.$$

This fraction is evidently continuous in the closed subset of D consisting of the origin plus those points where $g \geq 1$, and takes on the value one at the origin. Thus, since $(\nabla g)^2 > 0$ in $D - O$, it follows that

$$\tau^4(\nabla g)^2 > A > 0,$$

for all $\frac{1}{2} \geq \tau \geq 0$, where A is a positive number independent of τ and $P_0(x_0, y_0, z_0)$ on $g = 1$.

It then follows that there is a number B such that

$$\sum_{j=1}^{n-1} | \psi_i(\tau_{j+1}, x_0, y_0, z_0) - \psi_i(\tau_j, x_0, y_0, z_0) | < B,$$

independently of the partition and of $P_0(x_0, \cdots, z_0)$ on $g = 1$.

This completes the proof.

2.54. *Each trajectory has a forward tangent at the origin.*

To prove this we need to show that for each P_0 on $g=1$ the right hand derivatives $(\partial^+\psi_i/\partial\tau^+)_{\tau=0}$, or more simply $(\partial\psi_i/\partial\tau)_{\tau=0}$, exist and are not all zero. The forward tangent is then the directed half-line running from O through the point $[(\partial\psi_1/\partial\tau)_{\tau=0}, (\partial\psi_2/\partial\tau)_{\tau=0}, \partial\psi_3/\partial\tau)_{\tau=0}]$.

The proof of the existence is much the same in the three cases $i=1,2,3$. We need therefore consider only the first.

We have in dropping the x_0, y_0, z_0, for the moment, and representing by τ_1, τ_2 two numbers $\frac{1}{2} > \tau_2 > \tau_1 > 0$,

$$[\psi_1(\tau_2)/\tau_2] - [\psi_1(\tau_1)/\tau_1] = [\psi_1(\tau_2)/r_2] - [\psi_1(\tau_1)/r_1] + (1+v_2)\psi_1(\tau_2) - (1+v_1)\psi_1(\tau_1),$$

where r_j and v_j are the values of r and v corresponding to $\tau=\tau_j$. Thus

$$|\,[\psi_1(\tau_2)/\tau_2] - [\psi_1(\tau_1)/\tau_1]\,| \leq \left|\int_{\tau_1}^{\tau_2} [d(\psi_1/r)/d\tau]d\tau\right| + B_1\tau_2,$$

where B_1 is a number independent of τ_1, τ_2 and P_0. Accordingly,

$$\left|\frac{\psi_1(\tau_2)}{\tau_2} - \frac{\psi_1(\tau_1)}{\tau_1}\right| \leq \int_{\tau_1}^{\tau_2} \left|\frac{v_x - x/r(\nabla r)\cdot(\nabla v)}{(\nabla g)^2 r\tau^2}\right| d\tau + B_1\tau_2.$$

Now the quantity

$$|\,v_x - x/r(\nabla r)\cdot(\nabla v)\,| < C,$$

where C is independent of x, y, z in that subset of $D-O$ for which $g \geq 1$. Moreover,

$$(\nabla g)^2 r\tau^2 > Ar/\tau^2 = A/\tau(1-\tau-\tau v) > A'/\tau,$$

where A' is a positive number independent of $\tau \leq \frac{1}{2}$ and P_0 on $g=1$. Accordingly, there is a number B' independent of P_0 on $g=1$, $0<\tau_1<\tau_2<\frac{1}{2}$, such that

$$|\,\psi_1(\tau_2)/\tau_2 - \psi_1(\tau_1)/\tau_1\,| < B'\tau_2.$$

It follows that

$$\lim_{\tau=0}[\psi_1(\tau, x_0, y_0, z_0)/\tau]$$

exists, and exists, be it noted, uniformly with regard to P_0 on $g=1$. Thus $(\partial\psi_1/\partial\tau)_{t=0}$ exists.

That the three derivatives are not all zero at $\tau=0$ follows immediately from the relation

$$\sum_{i=1}^{3}[\psi_i(\tau, x_0, \cdots, z_0)/\tau]^2 = (1+rv+r)^2,$$

for the right-hand member of this equality approaches one as τ approaches zero.

2.55. *Every ray ρ issuing from O is the forward tangent of one and only one orthogonal trajectory.*

That there is one trajectory having the required property we pointed out in Section 2.4. To show there is at most one it is sufficient to demonstrate that, in using the notation of Sections 2.3, 2.4, if $C_{\bar{\alpha},\bar{\beta}}$ or $C_{\bar{\alpha}',\bar{\beta}'}$ has this property then no other trajectory has. Since the reasoning is the same in the two cases we need consider only the former.

We suppose that the trajectory T_{P_0} consisting of points $P_\tau'(x_\tau', y_\tau', z_\tau')$,

$$x_\tau' = \psi_1(\tau, x_0, y_0, z_0), \cdots, z_\tau' = \psi_3(\tau, x_0, y_0, z_0), \qquad 1 \geq \tau > 0,$$

is distinct from $C_{\bar{\alpha},\bar{\beta}}$ but has ρ as its tangent at O.

We choose $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\pi/8 < \alpha_1 < \bar{\alpha} < \alpha_2 < 7\pi/8, \ \pi/8 < \beta_1 < \bar{\beta} < \beta_2 < 15\pi/8.$$

We denote by γ_{τ_1} the set of points in Γ_1

$$\gamma_{\tau_1}: \quad \tau = \tau_1, \quad \alpha_1 \leq \alpha \leq \alpha_2, \quad \beta_1 \leq \beta \leq \beta_2,$$

and by D_{τ_1} the points of D corresponding to the points q of γ_{τ_1}. The points of D_{τ_1} are then points of the surface $g = 1/\tau_1 - 1$.

Now for sufficiently small values of τ, say $\tau \leq \bar{\tau}$, the surface $g = 1/\tau - 1$ is pierced by each ray issuing from O at one and only one point. Hence, in as much as the surface S, $g = 1/\bar{\tau} - 1$, consists of a closed set of points, it follows that S is homeomorphic with a sphere. Thus, since every level surface can be mapped in a 1:1 and continuous manner on S by means of the functions $f_i(t, x_0, y_0, z_0)$, every level surface is the homeomorph of a sphere.

Consider then the points E_{τ_1} of D_{τ_1} corresponding to the boundary points e_{τ_1} of γ_{τ_1}. The points E_{τ_1} form a simple closed curve consisting of four analytic arcs, and since the surface $g = 1/\tau_1 - 1$ is homeomorphic with a sphere, this curve divides it into two continua A_{τ_1}, B_{τ_1}, each having E_{τ_1} as its boundary. The set $D_{\tau_1} - E_{\tau_1}$ is certainly contained in one of these continua, say A_{τ_1}, for any two points in $D_{\tau_1} - E_{\tau_1}$ can be joined by a continuous curve not passing through E_{τ_1}. Moreover, every point of A_{τ_1} is a point of $D_{\tau_1} - E_{\tau_1}$. For if the contrary were true, there would be, because D_{τ_1} is closed, a point $P'(x', y', z')$,

$$x' = \phi_1(\tau_1, \alpha_1, \beta_1), \cdots, z' = \phi_3(\tau_1, \alpha_1, \beta_1),$$

contained in $D_{\tau_1} - E_{\tau_1}$, in every neighborhood of which would lie points of the surface $g = 1/\tau_1 - 1$ which were not points of D_{τ_1}. But this is readily

shown to be impossible in making use of the fact that the $\phi_i(\tau, \alpha, \beta)$ are analytic at $\tau_1, \alpha_1, \beta_1$ and have a non-vanishing Jacobian there.

Now P_{τ_1}' cannot belong to D_{τ_1} for any τ_1, for there is at most one trajectory through each point of D_{τ_1} and thus T_{P_0} would have to be one of the curves of the family $\{C_{\alpha,\beta}\}$. But every curve of this family distinct from $C_{\bar{\alpha},\bar{\beta}}$ has a forward tangent at O different from ρ.

Accordingly, P'_{τ_1} is in B_{τ_1} for every τ_1.

But this likewise is impossible. For we observe that the function

$$L(\tau_1, \alpha, \beta) = \sphericalangle P_{\tau_1} O \bar{P}_{\tau_1},$$

where P_{τ_1} is the point on D_{τ_1} corresponding to $q(\tau_1, \alpha, \beta)$ in γ_{τ_1} and $\bar{P}_{\tau_1}$ is the point of $C_{\bar{\alpha},\bar{\beta}}$ corresponding to $q(\tau_1, \bar{\alpha}, \bar{\beta})$, is continuous on the closed set e_{τ_1} and consequently takes on its minimum $\omega(\tau_1)$ there. If P_{τ_1}' were contained in B_{τ_1} we should have

$$\sphericalangle P_{\tau_1}' O \bar{P}_{\tau_1} > \omega(\tau_1)$$

when $\tau_1 \leq \bar{\tau}$, for otherwise we could pass from P_{τ_1}' to $\bar{P}_{\tau_1}$ by the curve formed from the intersection of the surface $g = 1/\tau_1 - 1$ with the plane determined by the rays $\vec{OP}'_{\tau_1}$, $\vec{OP}_{\tau_1}$. It would then follow that

$$\lim_{\tau_1=0} \omega(\tau_1) = 0, \tag{34}$$

for by hypothesis

$$\lim_{\tau_1=0} \sphericalangle P_{\tau_1}' O P_{\tau_1} = 0.$$

But, on the other hand,

$$\lim_{\tau_1=0} L(\tau_1, \alpha, \beta) = L(\alpha, \beta)$$

exists uniformly with regard to q on γ_{τ_1} (Section 2.54) and

$$L(\alpha, \beta) > 0$$

for all q on γ_{τ_1}. Consequently (34) cannot hold, and therefore P_{τ_1}' cannot be contained in B_{τ_1}.

Thus the property is demonstrated.

3.1. *The Mapping.* It is evident from the preceding properties of the trajectories that D can be mapped upon the interior D' of a sphere in such a way that the conditions 1 and 2 of the Introduction are satisfied. This may be done, in fact, by erecting in D' a set of rectangular axes x', y', z', with origin at O', the center of D', and taking as the image of the point P in D the point P' in D' lying on the sphere

$$x'^2 + y'^2 + z'^2 = \{\Delta/[1 + g(P)]\}^2,$$

where Δ is the radius of D', and lying on the radius having the same direction angles, referred to the x', y', z'-axes, as the forward tangent at O of the trajectory through P.

That this method of mapping, with the further definition that the image of O be O', satisfies the condition 3, that is, that it yields at 1 : 1 correspondence between the points of D and those of D', is not difficult to prove. In fact, on the one hand, since there is one and only one trajectory through each point of D, every such point has a unique image in D' while, on the other hand, every point P' in $D' - O'$ determines uniquely a number τ,

$$x'^2 + y'^2 + z'^2 = (\Delta\tau)^2,$$

and a radius ρ issuing from O, and thus since there is one and only one trajectory having ρ as its forward tangent at O, P' is the image of a unique point P in D.

It can be proved also that the condition 4 is satisfied, that is, that the functions

$$x' = h_1(x, y, z), \quad y' = h_2(x, y, z), \quad z' = h_3(x, y, z),$$

which give the co-ordinates of the point $P'(x', y', z')$, corresponding to the point $P(x, y, z)$, are analytic in $D - O$ and continuous at O.

To show that they are continuous at O we have only to observe that

$$\lim_{p \to O} (x'^2 + y'^2 + z'^2) = \lim_{p \to O} \{\Delta/[1 + g(P)]\}^2$$
$$= 0.$$

To show they are analytic in $D - O$ we observe firstly that when $\Delta\tau$, α, β are regarded as the spherical coördinates of $P'(x', y', z')$, the colatitude α being measured from the positive z'-axis and the longitude β from the positive x'-axis, counterclockwise with regard to the positive z'-axis, then the co-ordinates x, y, z of the point P are given by the formulas

$$x = \phi_1(\tau, \alpha, \beta), \quad y = \phi_2(\tau, \alpha, \beta), \quad z = \phi_3(\tau, \alpha, \beta),$$

provided that P' lies in the subset D_1' of D' determined by the inequalities

$$0 < \tau < 1, \quad \pi/8 < \alpha < 7\pi/8, \quad \pi/8 < \beta < 15\pi/8.$$

This follows from Section 2. 3.

Similarly, if $\Delta\tau$, α', β' are regarded as the spherical coördinates of P', the colatitude α' being measured in this case from the positive y'-axis and the longitude β' from the negative x'-axis, counterclockwise with regard to the positive y'-axis, then the coördinates of P are given by the formulas

$$x = \phi_1'(\tau, \alpha', \beta'), \qquad y = \phi_2'(\tau, \alpha', \beta'), \qquad z = \phi_3'(\tau, \alpha', \beta'),$$

provided that P' lies in the subset D_2' determined by the inequalities

$$0 < \tau < 1, \quad \pi/8 < \alpha' < 7\pi/8, \quad \pi/8 < \beta' < 15\pi/8.$$

Now since

$$D_1' + D_2' = D' - O$$

and τ, α, β are analytic functions of x', y', z' in D_1', and τ, α', β' are analytic functions of these variables in D_2', it follows that the functions

$$x = \bar{h}_1(x', y', z'), \quad y = \bar{h}_2(x', y', z), \quad z = \bar{h}_3(x', y', z'),$$

which give the coördinates of P in terms of those of P' are analytic in $D' - O'$. Moreover we find in employing (d) of Theorems II, II' that

$$\partial(\bar{h}_1, \bar{h}_2, \bar{h}_3) / \partial(x', y', z') = \Delta^3/(\nabla g)^2.$$

Accordingly, the functions h_i, which are the inverses of the $\bar{h}_i$, are analytic in $D - O$.

We summarize these results in the form of a theorem.

THEOREM III. *The region D can be mapped upon the interior D' of a sphere in such a way that the level surfaces of $g(D \mid O, P)$ are carried into spheres concentric with D' and the trajectories orthogonal to these surfaces are carried into radii of D'. When the mapping is done in the manner described above it is* 1:1 *and the coördinates x', y', z' of the point P' in D', corresponding to the point P in D are given by functions*

$$x' = h_1(x, y, z), \quad y' = h_2(x, y, z), \quad z' = h_3(x, y, z),$$

which are analytic in $D - O$ and continuous at O. These functions have the property that

$$\partial(h_1, h_2, h_3)/\partial(x, y, z) = (\nabla g)^2/\Delta^3.$$

3.2. We conclude this paper in observing that the transformations of Section 3.1 generalize completely conformal transformations in the sense that if a plane region is mapped upon the interior of a circle in an analogous manner and the relation between the level curves and their corresponding circles is properly specified then the transformation is necessarily conformal.

The Existence Theorems in the Problem of the Determination of Affine and Metric Spaces by their Differential Invariants.

By Tracy Yerkes Thomas.

Introduction. In a previous article we were concerned with the more formal aspects of the determination of affine and metric spaces by their tensor differential invariants.* We shall now take up the problem from its existence theoretic side. Let us first observe that if the components of a tensor, e. g. the components $A^i{}_{jk\alpha}$ or $g_{\alpha\beta,\gamma\delta}$ of a normal or metric tensor, are given originally as analytic functions of the $n(\geqq 2)$ independent variables x^i in the neighborhood of the point $x^i = q^i$, we can by the simple co-ordinate transformation $x^i = y^i + q^i$ transform them into components which are analytic functions of the variables y^i in the neighborhood of the point $y^i = 0$; in fact the direct substitution $x^i = y^i + q^i$ will transform each component $A^i{}_{jk\alpha}$ or $g_{\alpha\beta,\gamma\delta}$ with respect to the x^i variables into the corresponding component $A^i{}_{jk\alpha}$ or $g_{\alpha\beta,\gamma\delta}$ with respect to the y^i variables. We can therefore assume without loss of generality that the tensor components $A^i{}_{jk\alpha}$ or $g_{\alpha\beta,\gamma\delta}$ are analytic functions of the variables y^i in the neighborhood of the point $y^i = 0$. To illustrate more precisely the nature of the problem to be investigated let us consider a definite set of functions $A^i{}_{jk\alpha}(y)$ in $n(\geqq 2)$ independent variables y^i; each function $A^i{}_{jk\alpha}(y)$ is assumed to be analytic in the neighborhood of the point $y^i = 0$. We wish to investigate the question of whether there exists an affine connection defined by a set of functions $C^i{}_{jk}$ $(= C^i{}_{kj})$ each of which is analytic in the neighborhood of the point $y^i = 0$ such that the $A^i{}_{jk\alpha}(y)$ can be derived from the functions $C^i{}_{jk}(y)$ as the components of a normal tensor. An evident requirement is that the functions $A^i{}_{jk\alpha}(y)$ should satisfy the complete set of identities

$$\text{(a)} \qquad A^i{}_{jk\alpha} = A^i{}_{kj\alpha}; \qquad A^i{}_{jk\alpha} + A^i{}_{k\alpha j} + A^i{}_{\alpha jk} = 0.$$

In case $n = 2$ we shall show that these conditions are sufficient for the existence of the components $C^i{}_{jk}(y)$ of the affine connection; in fact if the conditions (a) are satisfied the connection $C^i{}_{jk}(y)$ is determined uniquely

* "Determination of affine and metric spaces by their differential invariants," *Mathematische Annalen*, Vol. 101 (1929), pp. 713-728. Reference to equations in this article will be made by prefixing the letter *M* to the number of the equation in question. The designation *M* will be used to refer to the article itself.

within a system of normal co-ordinates y^i. But if $n \geqq 3$ the restrictions imposed by (a) on the functions $A^i_{jka}(y)$ are *not* sufficient to insure the existence of the affine connection $C^i_{jk}(y)$. For $n \geqq 3$ it follows that subject to (a) certain of the functions $A^i_{jka}(y)$ can be selected as entirely arbitrary and that others are partially arbitrary in the sense that they can be considered to reduce to arbitrary analytic functions of a number $m (= 2, \cdots, n-1)$ of the independent variables y^i for the initial values $y^i = 0$ of the $n-m$ remaining variables. A definite rule can be given for the separation of the functions $A^i_{jka}(y)$ into entirely arbitrary and partially arbitrary functions; this separation being effected the components $C^i_{jk}(y)$ of the affine connection are uniquely determined with respect to a system of normal co-ordinates y^i. Analogous remarks apply to the problem of the determination of the functions $g_{\alpha\beta,\gamma\delta}(y)$ in order that there shall exist a symmetric fundamental tensor $h_{\alpha\beta}(y)$ with respect to which the $g_{\alpha\beta,\gamma\delta}(y)$ will be the components of a second extension or metric tensor. Thus the functions $g_{\alpha\beta,\gamma\delta}(y)$ must satisfy the conditions

$$\text{(b)} \qquad g_{\alpha\beta,\gamma\delta} = g_{\beta\alpha,\gamma\delta} = g_{\alpha\beta,\delta\gamma}; \qquad g_{\alpha\beta,\gamma\delta} + g_{\alpha\gamma,\delta\beta} + g_{\alpha\delta,\beta\gamma} = 0$$

which are sufficient to insure the existence of the tensor $h_{\alpha\beta}$ if $n = 2$ but not otherwise; also for $n \geqq 3$ certain of the functions $g_{\alpha\beta,\gamma\delta}$ are entirely arbitrary while certain others reduce to arbitrary functions of a number of the variables y^i for the initial values $y^i = 0$ of the remaining variables. Owing to the analogy between the affine and metric problems these two theories have been developed side by side.

From the standpoint of the theory of differential equations our problem, in the affine case, is to determine a set of analytic functions $A^i_{jka}(y)$ in the most general possible manner so that the two systems of equations (1.1) and (1.2) will be satisfied. We reduce this problem to the problem of finding a set of integrals A_{lm} (the designation A_{lm} is employed to denote certain components A^i_{jka} which remain independent after the conditions (a) have been imposed) of the system (6.1) in which (1) the independent variables y^i are the co-ordinates of a system of normal co-ordinates, and (2) the ★ terms denote polynomials in the components A_{lm} and C^i_{jk} such as result by taking the covariant derivative of the tensor A^i_{jka}. If the ★ terms were lacking in equations (6.1) these equations would break up into $n-2$ separate systems of equations each of which is of the type treated by König * and their integration could easily be accomplished. On account of

* J. König, "Ueber die Integration simultaner systeme partieller differentialgleichungen mit mehreren unbekannten Functionen," *Mathematische Annalen*, Vol. 23 (1884), pp. 520-526.

the fact that the derivatives of the components C^{i}_{jk} in the ★ terms in (6.1) must be related to the components A_{lm} and their derivatives in the process of determining the power series expansions of the $A_{lm}(y)$ we have an unusual situation. The more or less elaborate analysis employed has been necessitated by the fact that a situation of this type does not exist in any system of differential equations so far treated in the literature. A similar characterization of the metric problem is possible. In § 1 the conditions on the power series expansions of the functions A^{i}_{jka} and $g_{a\beta,\gamma\delta}$ have been given in order that these functions may be the components of a normal or metric tensor respectively. The equations giving these conditions as well as the convergence proofs in § 2 are of importance in the subsequent sections. The finite existence theorems for the functions A^{i}_{jka} and $g_{a\beta,\gamma\delta}$ are established in §§ 3, 5, and 6 on the basis of new forms in § 4 of the expressions $N(n, 1+r)$ which gives the number of independent components $A^{i}_{jka\beta_1\ldots\beta_r}$ and $N(n, 2+r)$ which gives the number of independent components $g_{a\beta,\gamma\delta\epsilon_1\ldots\epsilon_r}$ (see *M* § 1). We have in § 4 derived the new form of the expressions $N(n, p+r)$ for arbitrary values p which can be used in the extension of the results of the present article to normal and metric tensors of higher order. The convergence proofs involved in the finite existence theorems have not been given in the present article as they will appear in a slightly more general form in a later publication.*

1. *General Existence Conditions.* Let us consider that $A^{i}_{jka}(y)$ denotes a set of functions of the variable y^{i} each of which can be expanded about the point $y^{i}=0$ in a convergent power series; we shall also assume that the functions A^{i}_{jka} satisfy the equations (a). It is desired to identify the variables y^{i} with the co-ordinates of a system of normal co-ordinates and to choose the functions A^{i}_{jka} in such a way, by imposing further necessary conditions, that they will be the components of a normal tensor $A^{i}_{jka}(y)$ in this system of co-ordinates. This means that we wish to choose the functions $A^{i}_{jka}(y)$ so that the system of equations

$$(1.1)\qquad A^{i}_{jka}(y)=(\partial C^{i}_{jk}/\partial y^{a})-C^{i}_{jka}-C^{i}_{\mu k}C^{\mu}_{ja}-C^{i}_{j\mu}C^{\mu}_{ka}$$

will have a solution given by a set of functions C^{i}_{jk} $(=C^{i}_{kj})$ each of which

* Two notes based on the present article have already appeared in Vol. 15 of the *Proceedings of the National Academy of Sciences.* The first of these treated the system of equations formed by putting the contracted normal tensor $A^{i}_{a\beta i}$ equal to zero, and the second the system which constitutes the Einstein field equations for free space. A general theory of such systems of equations to be published later in the *Annals of Mathematics* under the title "Invariantive Systems of Partial Differential Equations," will contain the convergence proofs omitted in the present article.

is analytic in the neighborhood of the point $y^i = 0$, and such that the equations

$$C^i_{jk}(y)y^jy^k = 0 \tag{1.2}$$

are satisfied identically in the variables y^i. To this end we observe that the functions $A^i_{jka}(y)$ chosen initially so as to satisfy only equations (a) determine an infinite sequence of sets of constants

$$\text{I} \qquad (A^i_{jka\beta_1})_0; \quad (A^i_{jka\beta_1\beta_2})_0; \cdots,$$

as explained in M § 4. The constants I determine the formal series for the functions C^i_{jk}, namely

$$C^i_{jk} = (A^i_{jka})_0 y^a + (1/2!)(A^i_{jka\beta_1})_0 y^a y^{\beta_1} + \cdots, \tag{1.3}$$

where $(A^i_{jka})_0$ has the value $A^i_{jka}(0)$. The series (1.3) evidently satisfy (1.2) since the quantities $(A)_0$ satisfy the complete sets of identities M (1.13). The requirement that (1.3) shall constitute a formal solution of the differential equations (1.1) leads to other conditions on the functions $A^i_{jka}(y)$ than those given by (a). Thus by repeated differentiation of (1.1) and evaluation at the point $y^i = 0$ we obtain equations

$$\begin{aligned} A^i_{jka,\beta_1\ldots\beta_r}(0) = {} & (2/3)(A^i_{jka\beta_1\ldots\beta_r})_0 \\ & -(1/3)/(A^i_{kaj\beta_1\ldots\beta_r})_0 - (1/3)/(A^i_{ajk\beta_1\ldots\beta_r})_0 + \bigstar, \end{aligned} \tag{1.4}$$

where the $\bigstar$ represents an expression which is quadratic and homogeneous in the components of normal tensors of lower order than those which have been written down explicitly; also

$$A^i_{jka,\beta_1\ldots\beta_r}(0) = \left(\frac{\partial^r A^i_{jka}(y)}{\partial y^{\beta_1}\cdots\partial y^{\beta_r}}\right)_{y^i=0}.$$

In (1.4) the right members as well as the left are symmetric in the indices $\beta_1\cdots\beta_r$ arising from the differentiation; no new relations can therefore result from (1.4) as conditions of integrability. Now eliminate the above constants I which stand in the right members of (1.4) by a substitution of the type M (2.10). This will give a system of equations of the general form

$$\begin{aligned} A^i_{jka,\beta_1\ldots\beta_r}(0) = {} & L[A^i_{jka,\beta_1\ldots\beta_r}(0)] \\ & + B[A^i_{jka}(0); \cdots; A^i_{jka,\beta_1\ldots\beta_{r-2}}(0)], \end{aligned} \tag{1.5}$$

where as usual L has been used to denote a simple linear sum and B denotes a sum of homogeneous polynomials of its arguments. Equations (1.5) give conditions on the quantities $A^i_{jka,\beta_1\ldots\beta_r}(0)$. If the functions $A^i_{jka}(y)$ satisfy (a) and are furthermore such that the quantities $A^i_{jka,\beta_1\ldots\beta_r}(0)$ will satisfy

the sequence of equations (1.5) obtained by taking $r=1, 2, \cdots$, then (1.3) will constitute a unique formal solution of the equations (1.1) and (1.2). In § 2 it will be shown that the formal solution $C^i{}_{jk}(y)$ given by (1.3) converges. The equations (1.2) therefore give an actual characterization of the variables y^i as normal co-ordinates.

THEOREM A. *The set of functions $A^i{}_{jk\alpha}(y)$, each of which is analytic in the neighborhood of the point $y^i=0$, will be the components of a normal tensor in a system of normal co-ordinates y^i if, and only if, the functions $A^i{}_{jk\alpha}(y)$ satisfy (a) and are furthermore such that the coefficients $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}(0)$ of their power series expansions about the point $y^i=0$ satisfy the sequence of equations* (1.5).

The precise form of the equations (1.5) for $r=1$ can be obtained by eliminating the quantities $A^i{}_{jk\alpha\beta}$ in the right members of $M(2.1)$ by means of $M(2.8)$. This gives

$$3A^i{}_{jk\alpha,\beta}=3A^i{}_{jk\beta,\alpha}+2A^i{}_{j\beta\alpha,k}+2A^i{}_{k\beta\alpha,j}+A^i{}_{\alpha\beta j,k}+A^i{}_{\alpha\beta k,j} \tag{1.6}$$

as the conditions to be satisfied by the quantities

$$A^i{}_{jk\alpha,\beta}=A^i{}_{jk\alpha,\beta}(0).$$

Now consider a set of constants $(g_{\alpha\beta})_0=(g_{\beta\alpha})_0$, such that the determinant $|(g_{\alpha\beta})_0|\neq 0$, and a set of functions $g_{\alpha\beta,\gamma\delta}(y)$ each of which is analytic in the neighborhood of the point $y^i=0$. Let us state the conditions that must be satisfied by the functions $g_{\alpha\beta,\gamma\delta}(y)$ so that they will be the components of a metric tensor resulting from a fundamental tensor with symmetric components $h_{\alpha\beta}(y)$ in a system of normal co-ordinates y^i such that $h_{\alpha\beta}(0)=(g_{\alpha\beta})_0$ at the origin of the system. In other words we wish to find conditions on the functions $g_{\alpha\beta,\gamma\delta}(y)$ so that there will exist a solution of the system of differential equations

$$\begin{aligned} g_{\alpha\beta,\gamma\delta}(y)=&(\partial^2 h_{\alpha\beta}/\partial y^\gamma\partial y^\delta)-h_{\sigma\beta}C^\sigma{}_{\alpha\gamma\delta}-h_{\alpha\sigma}C^\sigma{}_{\beta\gamma\delta}-(\partial h_{\alpha\beta}/\partial y^\sigma)C^\sigma{}_{\gamma\delta}\\ &-(\partial h_{\sigma\beta}/\partial y^\gamma)C^\sigma{}_{\alpha\delta}-(\partial h_{\sigma\beta}/\partial y^\delta)C^\sigma{}_{\alpha\gamma}\\ &-(\partial h_{\alpha\sigma}/\partial y^\gamma)C^\sigma{}_{\beta\delta}-(\partial h_{\alpha\sigma}/\partial y^\delta)C^\sigma{}_{\beta\gamma}\\ &+h_{\sigma\tau}C^\sigma{}_{\alpha\gamma}C^\tau{}_{\beta\delta}+h_{\sigma\tau}C^\sigma{}_{\alpha\delta}C^\tau{}_{\beta\gamma} \end{aligned} \tag{1.7}$$

given by a set of functions $h_{\alpha\beta}$ $(=h_{\beta\alpha})$ each of which is analytic in the neighborhood of the point $y^i=0$, which are such that $h_{\alpha\beta}(0)=(g_{\alpha\beta})_0$, and which satisfy the equations

$$(\partial h_{\alpha\beta}/\partial y^\gamma)y^\beta y^\gamma=0 \tag{1.8}$$

identically in the variables y^i. These latter equations can be shown to be equivalent to (1. 2) in which the $C^i{}_{\alpha\beta}(y)$ are Christoffel symbols, i. e.

$$C^i{}_{\alpha\beta}(y) = \tfrac{1}{2}h^{i\sigma}[(\partial h_{\alpha\sigma}/\partial y^\beta) + (\partial h_{\sigma\beta}/\partial y^\alpha) - (\partial h_{\alpha\beta}/\partial y^\sigma)].$$

It is now expedient however to use (1. 8) rather than (1. 2) to characterize the variables y^i as normal co-ordinates. On the basis of the Christoffel symbols $C^i{}_{\alpha\beta}$ the quantities $C^i{}_{\alpha\beta\gamma}$ in (1. 7) as well as the quantities $C^i{}_{\alpha\beta\ldots\epsilon}$ in general which we shall need later are defined by the equations given in M § 1. We first assume that the functions $g_{\alpha\beta,\gamma\delta}(y)$ satisfy the complete set of identities (b) as this is a necessary condition. The process described in M § 4 will then determine the sets of constants

II $$(g_{\alpha\beta,\gamma\delta\epsilon_1})_0; \qquad (g_{\alpha\beta,\gamma\delta\epsilon_1\epsilon_2})_0; \cdots$$

in such a way that the formal series for $h_{\alpha\beta}$, namely

$$(1.9) \quad h_{\alpha\beta} = (g_{\alpha\beta})_0 + (1/2!)(g_{\alpha\beta,\gamma\delta})_0 y^\gamma y^\delta + (1/3!)(g_{\alpha\beta,\gamma\delta\epsilon_1})_0 y^\gamma y^\delta y^{\epsilon_1} + \cdots,$$

will satisfy (1. 8); in the series (1. 9) the constant $(g_{\alpha\beta,\gamma\delta})_0$ has the value $g_{\alpha\beta,\gamma\delta}(0)$. Now differentiate (1. 7) repeatedly, evaluate at the point $y^i = 0$ and eliminate the quantities $(g_{\alpha\beta,\gamma\delta\epsilon_1\ldots\epsilon_r})_0$ in the right members of the resulting equations by a substitution of the type $M(2.16)$. This gives conditions analogous to (1. 5), namely

$$(1.10) \quad g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0) = L[g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)] + Q[(g_{\alpha\beta})_0; g_{\alpha\beta,\gamma\delta}(0); \cdots; g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_{r-2}}(0)],$$

which must be satisfied by the quantities

$$g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0) = \left(\frac{\partial^r g_{\alpha\beta,\gamma\delta}(y)}{\partial y^{\epsilon_1}\cdots\partial y^{\epsilon_r}}\right)_{y^i=0}$$

in order that the series (1. 9) should constitute a formal solution (1. 7). As it will be shown in § 2 that this formal solution (1. 8) converges we have the following theorem.

THEOREM B. *Given a set of constants* $(g_{\alpha\beta})_0 = (g_{\beta\alpha})_0$, *such that the determinant* $|(g_{\alpha\beta})_0| \neq 0$, *and a set of functions* $g_{\alpha\beta,\gamma\delta}(y)$ *each of which is analytic in the neighborhood of the point* $y^i = 0$. *There will then exist a fundamental tensor having components* $h_{\alpha\beta}(y) = h_{\beta\alpha}(y)$ *in a system of normal co-ordinates* y^i *such that* (1) $h_{\alpha\beta}(0) = (g_{\alpha\beta})_0$, *and* (2) *the functions* $g_{\alpha\beta,\gamma\delta}(y)$ *are derivable from the* $h_{\alpha\beta}(y)$ *as the components of a metric tensor if, and only if, the functions* $g_{\alpha\beta,\gamma\delta}(y)$ *satisfy equations* (b) *and are furthermore*

such that the coefficients $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ *of their power series expansions about the point* $y^i = 0$ *satisfy the sequence of equations* (1.10).

To deduce equations corresponding to (1.6) differentiate (1.7) and evaluate at the point $y^i = 0$. This gives

$$6g_{\alpha\beta,\gamma\delta,\epsilon} = 4g_{\alpha\beta,\gamma\delta\epsilon} + g_{\alpha\epsilon,\beta\gamma\delta} + g_{\beta\epsilon,\alpha\gamma\delta} + 2g_{\gamma\delta,\alpha\beta\epsilon}, \tag{1.11}$$

where $g_{\alpha\beta,\gamma\delta\epsilon}$ denotes the third derivative of $h_{\alpha\beta}$ with respect to y^γ, y^δ, y^ϵ evaluated at the point $y^i = 0$, and

$$g_{\alpha\beta,\gamma\delta,\epsilon} = \left(\frac{\partial g_{\alpha\beta,\gamma\delta}(y)}{\partial y^\epsilon}\right)_{y^i=0}.$$

Eliminating the quantities $g_{\alpha\beta,\gamma\delta\epsilon}$ in the right members of (1.10) by $M(2.18)$ we deduce

$$2g_{\alpha\beta,\gamma\delta,\epsilon} = g_{\alpha\beta,\delta\epsilon,\gamma} + g_{\alpha\beta,\gamma\epsilon,\delta} + g_{\beta\epsilon,\gamma\delta,\alpha} + g_{\alpha\epsilon,\gamma\delta,\beta}. \tag{1.12}$$

Equations (1.6) and (1.12) can also be considered to give relations between the covariant derivatives $A^i{}_{jk\alpha,\beta}$ and $g_{\alpha\beta,\gamma\delta,\epsilon}$ respectively which must hold throughout the neighborhood of the origin of the normal co-ordinate system. This more general interpretation of (1.6) and (1.12) will be of importance later.

2. *Convergence proofs.* A proof of the convergence of the series (1.3) and (1.9) can be made by the method involving the use of dominant functions, i. e. the *Calcul des limites* of Cauchy. Let us first prove the convergence of the series (1.3) where it is to be understood that this series is determined as above explained by a set of functions $A^i{}_{jk\alpha}(y)$ which satisfy the conditions stated in Theorem A. It can be shown readily that the function $\mathfrak{C}^i{}_{jk}$ defined as a solution of the system of differential equations

$$\partial\mathfrak{C}^i{}_{jk}/\partial y^\alpha = 4\mathfrak{A}^i{}_{jk\alpha} + 2(\mathfrak{C}^i{}_{\mu k}\mathfrak{C}^\mu{}_{j\alpha} + \mathfrak{C}^i{}_{\mu\alpha}\mathfrak{C}^\mu{}_{jk}) \tag{2.1}$$

such that $\mathfrak{C}^i{}_{jk} = 0$ when $y^1 = \cdots = y^n = 0$, is a dominant function for the connection $C^i{}_{jk}$ given by (1.3); in (2.1) the function $\mathfrak{A}^i{}_{jk\alpha}$ is a dominant function for $A^i{}_{jk\alpha}(y)$. Equations (2.1) can in fact be reduced to a very simple form by putting

$$\mathfrak{C}^i{}_{jk} = \Phi; \qquad \mathfrak{A}^i{}_{jk\alpha} = F,$$

where F, defined by the equation

$$F = \frac{M}{1-(y^1+\cdots+y^n)/\rho} \tag{2.2}$$

for suitable positive constants M and ρ, is a dominant function for any of the functions $A^i{}_{jk\alpha}$. With the above substitutions (2.1) becomes

$$d\Phi/dy = 4F + 4n\Phi^2, \tag{2.3}$$

where $y = y^1 + \cdots + y^n$. The existence of a solution Φ of the equation (2.3) which is analytic in the neighborhood of $y = 0$, and such that $\Phi = 0$ for $y = 0$, results from the well known theorem of partial differential equations. To show that the function Φ satisfying these initial conditions dominates the series (1.3) for the affine connection $C^i{}_{jk}$ let us differentiate (2.1) repeatedly and evaluate the resulting equations at the point $y^i = 0$. We thus obtain expressions of the general form

$$\begin{aligned}\Phi^i{}_{jk\alpha\beta_1\ldots\beta_r} &= 4F^i{}_{jk\alpha\beta_1\ldots\beta_r} \\ &\quad + 2(\partial^r/\partial y^{\beta_1}\cdots\partial y^{\beta_r})(\mathfrak{C}^i{}_{\mu k}\mathfrak{C}^\mu{}_{j\alpha} + \mathfrak{C}^i{}_{\mu\alpha}\mathfrak{C}^\mu{}_{jk})_{y^i=0},\end{aligned} \tag{2.4}$$

where $\Phi^i{}_{jk\alpha\beta_1\ldots\beta_r}$ and $F^i{}_{jk\alpha\beta_1\ldots\beta_r}$ denote the derivatives of $\partial\mathfrak{C}^i{}_{jk}/\partial y^\alpha$ and $\mathfrak{A}^i{}_{jk\alpha}$ with respect to $y^{\beta_1}, \cdots, y^{\beta_r}$ evaluated at $y^i = 0$. Assuming that

$$\Phi^i{}_{jk\alpha\beta_1\ldots\beta_s} > |\,(A^i{}_{jk\alpha\beta_1\ldots\beta_s})_0\,| \tag{2.5}$$

for $s < r(\geqq 2)$ we can show that for $s = r$ these inequalities are likewise satisfied. Compare equations (2.4) with equations

$$A^i{}_{jk\alpha\beta_1\ldots\beta_r} = A^i{}_{j\alpha k\beta_1\ldots\beta_r} + A^i{}_{jk\alpha,\beta_1\ldots\beta_r} - A^i{}_{j\alpha k,\beta_1\ldots\beta_r} + \bigstar, \tag{2.6}$$

(see M § 2) in which $A^i{}_{jk\alpha}$ in the ★ term has the value $A^i{}_{jk\alpha}(0)$ and the quantities $A^i{}_{jk\alpha\beta_1\ldots\beta_r}$ and $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}$ are considered to represent the constants $(A^i{}_{jk\alpha\beta_1\ldots\beta_r})_0$ and $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}(0)$ for any value of $r(\geqq 1)$. This comparison shows that

$$2(\partial^r/\partial y^{\beta_1}\cdots\partial y^{\beta_r})(\mathfrak{C}^i{}_{\mu k}\mathfrak{C}^\mu{}_{j\alpha} + \mathfrak{C}^i{}_{\mu\alpha}\mathfrak{C}^\mu{}_{jk})_{y^i=0}$$

is greater than twice the absolute value of the ★ terms in (2.6) in virtue of (2.5) for $s < r$; also $4F^i{}_{jk\alpha\beta_1\ldots\beta_r}$ is greater than twice the absolute value of the difference $A^i{}_{jk\alpha,\beta_1\ldots\beta_r} - A^i{}_{j\alpha k,\beta_1\ldots\beta_r}$ in (2.6) because of the dominating property of the function F. Hence

$$\Phi^i{}_{jk\alpha\beta_1\ldots\beta_r} > 2\,|\,(A^i{}_{jk\alpha\beta_1\ldots\beta_r})_0 - (A^i{}_{j\alpha k\beta_1\ldots\beta_r})_0\,|\,. \tag{2.7}$$

Since the $\Phi^i{}_{jk\alpha\beta_1\ldots\beta_r}$ are symmetric with respect to all indices we also have

$$\Phi^i{}_{jk\alpha\beta_1\ldots\beta_r} > 2\,|\,(A^i{}_{j\alpha k\beta_1\ldots\beta_r})_0 - (A^i{}_{\alpha\beta_1 jk\ldots\beta_r})_0\,|\,. \tag{2.8}$$

Adding (2.7) and (2.8) we obtain

$$\Phi^i{}_{jk\alpha\beta_1\ldots\beta_r} > |\,(A^i{}_{jk\alpha\beta_1\ldots\beta_r})_0 - (A^i{}_{\alpha\beta_1 jk\ldots\beta_r})_0\,|\,.$$

Hence $\Phi^i{}_{jk\alpha\beta_1\dots\beta_r}$ is greater than the absolute value of the difference of $(A^i{}_{jk\alpha\beta_1\dots\beta_r})_0$ and $(A^i{}_{pq\sigma\tau_1\dots\tau_r})_0$ where $pq\sigma\tau_1\cdot\cdot\cdot\tau_r$ is any permutation of $jk\alpha\beta_1\cdot\cdot\cdot\beta_r$; we can therefore write

$$\Phi^i{}_{jk\alpha\beta_1\dots\beta_r} > |\,(A^i{}_{jk\alpha\beta_1\dots\beta_r})_0 - (A^i{}_{pq\sigma\tau_1\dots\tau_r})_0\,|\,. \tag{2.9}$$

Now form all permutations of the indices $pq\sigma\tau_1\cdot\cdot\cdot\tau_r$ and add together the $\mathfrak{N}$ corresponding inequalities (2.9) obtaining

$$\mathfrak{N}\Phi^i{}_{jk\alpha\beta_1\dots\beta_r} > |\,\mathfrak{N}(A^i{}_{jk\alpha\beta_1\dots\beta_r})_0 - \Sigma(A^i{}_{pq\sigma\tau_1\dots\tau_r})_0\,|\,.$$

But the sum in the right member of this inequality vanishes by $M(1.13)$. Hence we obtain the inequality (2.5) for $s = r(\geqq 2)$ which we wished to prove. The fact that (2.5) holds for $s = 1$ is proved directly by observing that for this case the ★ terms in (2.6) and the second set of terms in (2.4) vanish; we are thus led to the inequality (2.7) for $r = 1$ and hence to (2.5) for $s = 1$. The function $\mathfrak{G}^i{}_{jk}$ therefore dominates the series (1.3) for the affine connection $C^i{}_{jk}$ with the result that (1.3) must converge.

To prove the convergence of the series (1.9) we set up the system of differential equations

$$\partial^2\mathfrak{H}_{\alpha\beta}/\partial y^\gamma\partial y^\delta = 4\mathfrak{G}_{\alpha\beta\gamma\delta} + 4\sum[(\partial\mathfrak{H}_{\sigma\alpha}/\partial y^\beta)\mathfrak{N}^\sigma{}_{\gamma\delta} + \mathfrak{H}_{\sigma\tau}\mathfrak{N}^\sigma{}_{\alpha\beta}\mathfrak{N}^\tau{}_{\gamma\delta}], \tag{2.10}$$

where $\mathfrak{G}_{\alpha\beta\gamma\delta}$ is a dominant function for $g_{\alpha\beta,\gamma\delta}(y)$ and the summation Σ denotes the sum of all terms that can be formed from the term

$$(\partial\mathfrak{H}_{\sigma\alpha}/\partial y^\beta)\mathfrak{N}^\sigma{}_{\gamma\delta} + \mathfrak{H}_{\sigma\tau}\mathfrak{N}^\sigma{}_{\alpha\beta}\mathfrak{N}^\tau{}_{\gamma\delta} \tag{2.11}$$

by taking all permutations of the indices $\alpha\beta\gamma\delta$. Also the expression $\mathfrak{N}^i{}_{\alpha\beta}$ in the term (2.11) is defined by

$$\mathfrak{N}^i{}_{\alpha\beta} = \mathfrak{L}^{i\sigma}[(\partial\mathfrak{H}_{\sigma\beta}/\partial y^\alpha) + (\partial\mathfrak{H}_{\sigma\alpha}/\partial y^\beta) + (\partial\mathfrak{H}_{\alpha\beta}/\partial y^\sigma)],$$

where $\mathfrak{L}^{i\sigma}$ is a dominant function for $\mathfrak{H}^{i\sigma}$. Now replace $\mathfrak{H}_{\alpha\beta}$ by Ψ and take $\Psi_0 > |\,(g_{\alpha\beta})_0\,|$ (absolute value); also take

$$\mathfrak{G}_{\alpha\beta\gamma\delta} = \frac{M}{1-(y/\rho)}; \qquad \mathfrak{L}^{i\sigma} = \frac{M'}{1-(\Psi-\Psi_0)/\rho'},$$

where $y = y^1 + \cdot\cdot\cdot + y^n$ and M, M', ρ, ρ' are suitable positive constants. The equations (2.10) will then reduce to a single differential equation as in the case previously considered. This differential equation possesses a unique solution $\Psi(y)$ expandable in a convergent power series about $y = 0$

such that $\Psi(0)=\Psi_0$ and $d\Psi(y)/dy=0$ for $y=0$. The function $\Psi(y)$ so determined dominates the series (1.9). To show this we assume that

$$\Psi_{\alpha\beta\gamma\delta\epsilon_1\dots\epsilon_s} > |\,(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_s})_0\,| \tag{2.12}$$

for $s<r(\geqq 2)$ where the quantity $\Psi_{\alpha\beta\gamma\delta\epsilon_1\dots\epsilon_s}$ is equal to the derivative of $\mathfrak{H}_{\alpha\beta}$ with respect to $y^\gamma\cdots y^{\epsilon_s}$ evaluated at $y^i=0$. We shall show that (2.12) is also true for $s=r$ which will prove the dominating property of the function $\Psi(y)$. Due to the fact that $\mathfrak{H}_{\alpha\beta}$ is to be taken equal to the function $\Psi(y)$ it follows that $\Psi_{\alpha\beta\gamma\delta\epsilon_1\dots\epsilon_s}$ is symmetric with respect to all its indices. To prove (2.12) for $s=r$ we differentiate (2.10) repeatedly r times, then evaluate at $y^i=0$ and compare the resulting equations with the equations

$$\begin{aligned} &g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r}+g_{\delta\gamma,\beta\alpha\epsilon_1\dots\epsilon_r}-g_{\delta\beta,\gamma\alpha\epsilon_1\dots\epsilon_r}\\ &=g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta,\epsilon_1\dots\epsilon_r}+g_{\delta\gamma,\beta\alpha,\epsilon_1\dots\epsilon_r}-g_{\delta\beta,\gamma\alpha,\epsilon_1\dots\epsilon_r}+\bigstar \end{aligned} \tag{2.13}$$

[equations $M(2.14)$], in which the $g_{\alpha\beta,\gamma\delta}$ appearing in the $\bigstar$ terms has the value $g_{\alpha\beta,\gamma\delta}(0)$ and the quantities $g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}$ and $g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}$ have the values $(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r})_0$ and $g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}(0)$ respectively for any value of $r(\geqq 1)$. It is clear that

$$\begin{aligned} \Psi_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r} > |\,&(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r})_0-(g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r})_0\\ &+(g_{\gamma\delta,\beta\alpha\epsilon_1\dots\epsilon_r})_0-(g_{\delta\beta,\gamma\alpha\epsilon_1\dots\epsilon_r})_0\,|\,. \end{aligned} \tag{2.14}$$

We now introduce the operations P and P' defined in M § 2. It will be recalled that the operation P on a term such as $g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}$ effects the summation of all terms obtainable from $g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}$ by permutting the indices $\delta\epsilon_1\cdots\epsilon_r$ cyclically while the remaining indices $\alpha\beta\gamma$ are held fixed; similarly the operation P' on the term $g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}$ effects the summation of the terms obtainable from $g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}$ by taking all permutations of the indices $\gamma\delta\epsilon_1\cdots\epsilon_r$. First operating on both members of (2.14) by P we have

$$(r+1)\Psi_{\alpha\beta\gamma\delta\epsilon_1\dots\epsilon_r}>(r+2)\,|\,(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r})_0-(g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r})_0\,|\,.$$

We next operate on both members of these latter inequalities by P' which gives

$$(r+1)\Psi_{\alpha\beta\gamma\delta\epsilon_1\dots\epsilon_r}>(r+3)\,|\,(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r})_0\,|\,.$$

This proves (2.12) for $s=r(\geqq 2)$. If $s=1$ we obtain the inequality (2.14) for $r=1$ since the $\bigstar$ terms in (2.10) and (2.13) then vanish; operating on (2.14) for $r=1$ as above we obtain (2.12) for $s=1$. The function $\Psi(y)$ is therefore a dominant function for the series (1.9) and therefore the series (1.9) converges within a sufficiently small neighborhood of the point $y^i=0$.

3. *Special Case of Two Variables.* It is of interest to investigate the question of whether the conditions given by (1. 5) and (1. 10) on the derivatives of the functions $A^i{}_{jk\alpha}(y)$ and $g_{\alpha\beta,\gamma\delta}(y)$ at $y^i = 0$ are independent of those obtainable from the complete sets of identities of these functions, i. e. conditions of the type $M(1.15)$ and $M(1.14)$. The simplest case to consider is that of the functions $A^i{}_{jk\alpha}(y)$ or $g_{\alpha\beta,\gamma\delta}(y)$ for $n = 2$, i. e. for two variables. First take the functions $A^i{}_{jk\alpha}(y)$ and suppose that they satisfy the complete set of identities (a). The number $N(2, 1)$ defined in M § 1, and which is equal to 4, gives the number of independent components $A^i{}_{jk\alpha}$. Now if (a) gave all the conditions on the function $A^i{}_{jk\alpha}(y)$, the number of independent components of the extension $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}$ would be $4K(2, r)$ or $4(r+1)$. But actually the number of independent quantities $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}$ is given by $N(2, r+1)$* which is equal to $4(r+1)$. Hence it follows that the equation (1. 5) for $n = 2$ can not produce additional conditions on the quantities $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}$ over those which result from (a) by extension.

There is of course an analogous result for the functions $g_{\alpha\beta,\gamma\delta}(y)$ in two variables. Here the number of independent components $g_{\alpha\beta,\gamma\delta}(y)$ is $N(2, 2)$ or 1. Then, since $K(2, r)$ and $N(2, r+2)$ each have the value $r+1$, it follows that (b) gives all the functional conditions on the components $g_{\alpha\beta,\gamma\delta}(y)$; the conditions (1. 10) for $n = 2$ must therefore be satisfied identically in this case as a consequence of (b) and the conditions obtainable from (b) by extension.

THEOREM C. *For $n = 2$ the functions $A^i{}_{jk\alpha}(y)$, each of which is analytic in the neighborhood of the point $y^i = 0$, will be the components of a normal tensor in a system of normal co-ordinates if, and only if, the functions*

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* Since the components $A^i{}_{jk\alpha,\beta}$ are expressible in terms of the components $A^i{}_{jk\alpha\beta}$ by $M(2.1)$ and the components $A^i{}_{jk\alpha\beta}$ are expressible in terms of the components $A^i{}_{jk\alpha,\beta}$ by $M(2.8)$, it follows that the number of independent components $A^i{}_{jk\alpha,\beta}$ is equal to the number of independent components $A^i{}_{jk\alpha\beta}$. More generally if we consider the values of the components $A^i{}_{jk\alpha\beta_1\ldots\beta_s}$ and $A^i{}_{jk\alpha,\beta_1\ldots\beta_s}$ for $s < r$ to be determined then the number of independent components $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}$ is equal to the number of independent components $A^i{}_{jk\alpha\beta_1\ldots\beta_r}$ on account of the equations

$$A^i{}_{jk\alpha,\beta_1\ldots\beta_r} = A^i{}_{jk\alpha\beta_1\ldots\beta_r} - A^i{}_{(jk\alpha)\beta_1\ldots\beta_r} + \bigstar$$

(see M § 2) and the equations $M(2.10)$. Similarly the number of independent components $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}$ is equal to the number of independent components $g_{\alpha\beta,\gamma\delta\epsilon_1\ldots\epsilon_r}$ when the values (1) of the components $g_{\alpha\beta}$, and (2) of the components $g_{\alpha\beta,\gamma\delta\epsilon_1\ldots\epsilon_s}$ and $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_s}$ for $s < r$ are previously determined. In this sense we can say that the number of independent components $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}$ is $N(n, 1+r)$ as given by (4. 12) and that the number of independent components $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}$ is $N(n, 2+r)$ as given by (4. 13) in § 4.

$A^i_{jka}(y)$ satisfy (a). Similarly, for $n = 2$ the functions $g_{\alpha\beta,\gamma\delta}(y)$, each of which is analytic in the neighborhood of the point $y^i = 0$, will be the components of a metric tensor in normal co-ordinates if, and only if, the functions $g_{\alpha\beta,\gamma\delta}$ satisfy (b).

With the exception of the two cases treated above equations (1.5) and (1.10) furnish additional conditions over those which result by extension from the complete sets of identities (a) and (b) respectively. The determination of the exact arbitrariness of the functions $A^i_{jka}(y)$ or $g_{\alpha\beta,\gamma\delta}(y)$ in finite form for $n \geqq 3$ can be made on the basis of the numbers $N(n, p)$ by an extension of the above method. For this purpose we must derive a new form for the expressions $N(n, p)$.

4. *A new Form of the Expressions $N(n, p)$.* The formulas for $N(n, p)$ which determine the number of independent components $A^i_{jka_1 \ldots a}$ or $g_{\alpha\beta,\gamma_1 \ldots \gamma_p}$, given in *M* § 1, were originally derived from equations

Affine case: $N(n, p) = nK(n, 2)\, K(n, p) - nK(n, p+2), \quad p \geqq 1$

Metric case: $N(n, p) = K(n, 2)K(n, p) - nK(n, p+1), \quad p \geqq 2,$

where $K(n, p)$ denotes the number of combinations with repetitions of n things taken p at a time.* Since in addition to the above expressions for $N(n, p)$ we will need to have the expressions for $N(n, p+r)$ before us we note that

(4.1) Affine case: $N(n, p+r) = nK(n, 2)K(n, p+r) - nK(n, p+r+2)$

(4.2) Metric case: $N(n, p+r) = K(n, 2)K(n, p+r) - nK(n, p+r+1).$

Let us now determine numbers A_i and B_i depending only on n and p such that the above expressions for $N(n, p+r)$ can be written

(4.3) Affine case: $N(n, p+r) = \sum_{i=0}^{n-1} A_i K(n-i, r)$

(4.4) Metric case: $N(n, p+r) = \sum_{i=0}^{n-1} B_i K(n-i, r)$

for all integer values r. To derive (4.3) and (4.4) we make use of the formula

$$K(n, p) = K(n, p-1) + K(n-1, p)$$

by repeated application of which we have

$$K(n, r+1) = K(n, r) + K(n-1, r) + \cdots + K(2, r) + 1,$$

* See *Annals of Mathematics*, Vol. 28 (1927), pp. 196-236 and *ibid.*, pp. 631-688.

or

$$(4.5)\qquad K(n, r+1) = \sum_{i=0}^{n-1} K(n-i, r) = \sum_{\alpha=1}^{n} K(\alpha, r).$$

Hence

$$\begin{aligned} K(n, p+r) &= \sum_{\alpha=0}^{n-1} K(n-\alpha, p+r-1) \\ &= \sum_{\alpha=0}^{n-1} \sum_{\beta=0}^{n-\alpha-1} K(n-\alpha-\beta, p+r-2) \\ &\quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ &= \sum_{\alpha=0}^{n-1} \cdots \sum_{\epsilon=0}^{n-\alpha-\ldots-\eta-1} K(n-\alpha-\cdots-\eta-\epsilon, r), \end{aligned}$$

where the indices $\alpha\beta\cdots\eta\epsilon$ are p in number. Each of the indices $\alpha\beta\cdots\eta\epsilon$ can take on values $0, 1, \cdots, n-1$, subject to the condition that the sum $\alpha+\beta+\cdots+\eta+\epsilon$ has a value between zero and $n-1$ inclusive. The number of times that the sum $\alpha+\beta+\cdots+\eta+\epsilon$ takes on the value $i\ (=0, 1, \cdots, n-1)$ is equal to the number of permutations with repetitions of the numbers $0, 1, \cdots, i$ taken p at a time such that the sum of the numbers in any permutation is equal to i; let us denote the number of times that the sum $\alpha+\beta+\cdots+\eta+\epsilon$ takes on the value $i\ (=0, 1, \cdots, n-1)$ by $S(i+1, p-1)$. More generally $S(q+1, r-1)$ will denote the number of permutations with repetitions of $0, 1, \cdots, i$ taken r at a time such that the sum of the numbers in any permutation is equal to $q(\leqq i)$. Now observe that $S(q+1, r-1)$ is also equal to the number of permutations with repetitions of $0, 1, \cdots, i$ taken $r-1$ at a time such that the sum of the numbers in any permutation is $\leqq q$ for $q \leqq i$; this is seen immediately on account of the fact that from any permutation of this latter type one of the type which go to make up the sum $S(q+1, r-1)$ can be obtained by the addition of one of the numbers $0, 1, \cdots, i$, and the fact that in this way all permutations which make up the sum $S(q+1, r-1)$ can be constructed. Hence

$$S(q+1, r-1) = \sum_{\alpha=1}^{q+1} S(\alpha, r-2)$$

for $q \leqq i$. Using this formula as a recurrence formula we have

$$(4.6)\qquad S(i+1, p-1) = \sum_{\alpha=1}^{i+1} \sum_{\beta=1}^{\alpha} \cdots \sum_{\mu=1}^{\nu} S(\mu, 1).$$

But from (4.5) an exactly similar expression can be obtained for $K(i+1, p-1)$ in which the $S(\mu, 1)$ in (4.6) is replaced by $K(\mu, 1)$. Since $S(\mu, 1)$ and $K(\mu, 1)$ are each equal to μ it follows that $S(i+1, p-1)$ is equal

to $K(i+1, p-1)$. The above expression for $K(n, p+r)$ can therefore be written

$$K(n, p+r) = \sum_{i=0}^{n-1} K(i+1, p-1)\, K(n-i, r). \tag{4.7}$$

Substituting this expression for $K(n, p+r)$ and a similar expression for $K(n, p+r+2)$ into (4.1) we obtain (4.3), where

$$A_i = nK(n, 2)\, K(i+1, p-1) - nK(i+1, p+1). \tag{4.8}$$

Similarly by making a substitution of the type (4.7) into (4.2) we deduce (4.4), where

$$B_i = K(n, 2)\, K(i+1, p-1) - nK(i+1, p). \tag{4.9}$$

Let us also observe that

$$\text{Affine case:} \qquad N(n, p) = \sum_{i=0}^{n-1} A_i \tag{4.10}$$

$$\text{Metric case:} \qquad N(n, p) = \sum_{i=0}^{n-1} B_i. \tag{4.11}$$

These equations follow from (4.8) and (4.9) when use is made of (4.5).

In the following work we shall only need (4.3) for $p = 1$ and (4.4) for $p = 2$, i. e.

$$\text{Affine case:} \qquad N(n, 1+r) = \sum_{i=0}^{n-2} A_i K(n-i, r) \tag{4.12}$$

$$\text{Metric case:} \qquad N(n, 2+r) = \sum_{i=0}^{n-2} B_i K(n-i, r). \tag{4.13}$$

For these cases A_i and B_i are given by the special formulas

$$A_i = nK(n, 2) - nK(i+1, 2) \qquad (p=1) \tag{4.14}$$

$$B_i = K(n, 2)\, K(i+1, 1) - nK(i+1, 2) \qquad (p=2). \tag{4.15}$$

It is to be noted that $A_{n-1} = 0$ and $B_{n-1} = 0$ is a consequence of these latter formulas.

5. *Special Case of three Variables.* Let us derive in detail the existence theorems for the functions $A^i{}_{jka}(y)$ and $g_{a\beta,\gamma\delta}(y)$ for the case of $n = 3$ before entering upon the more general discussion in § 6. Now divide the 24 $(= N(3, 1))$ independent components $A^i{}_{jka}$ into a group G_0 consisting of 15 $(= A_0)$ components and a group G_1 consisting of 9 $(= A_1)$ components; similarly divide the 6 $(= N(3, 2))$ independent components $g_{a\beta,\gamma\delta}$ into a

group G_0 consisting of 3 ($= B_0$) components and a group G_1 consisting of 3 ($= B_1$) components. The following tables give these groups

Affine case: Group G_0				
a^i	b^i	c^i	d^i	e^i
A^i_{121}	A^i_{131}	A^i_{221}	A^i_{231}	A^i_{331}

Affine case: Group G_1		
u^i	v^i	w^i
A^i_{132}	A^i_{232}	A^i_{332}

Metric case: Group G_0		
α	β	γ
$g_{11,22}$	$g_{11,23}$	$g_{11,33}$

Metric case: Group G_1		
ζ	η	θ
$g_{21,23}$	$g_{21,33}$	$g_{22,33}$

in which the letters $a^i, b^i, \cdots, w^i$ and $\alpha, \cdots, \theta$ have been used as abbreviations for the components $A^i{}_{jk\alpha}$ and $g_{\alpha\beta,\gamma\delta}$. Now give the index β the value 1 and the indices $jk\alpha$ in equations (1.6) the values (132), (232), (332) respectively as in Group G_1 for the affine case; similarly put $\epsilon = 1$ and give the indices $\alpha\beta\gamma\delta$ in (1.12) the values (2131), (2133), (2233) as in group G_1 for the metric case. We thus arrive at the following two systems of equations

$$\begin{aligned} 2(\partial u^i/\partial y^1) &= -(\partial d^i/\partial y^1) + 3(\partial b^i/\partial y^2) - 3(\partial a^i/\partial y^3) + \bigstar \\ 6(\partial v^i/\partial y^1) &= 2(\partial u^i/\partial y^2) + 4(\partial d^i/\partial y^2) - 3(\partial c^i/\partial y^3) + \bigstar \\ 3(\partial w^i/\partial y^1) &= 3(\partial e^i/\partial y^2) + 2(\partial u^i/\partial y^3) - 2(\partial d^i/\partial y^3) + \bigstar \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} 2(\partial\zeta/\partial y^1) &= (\partial\beta/\partial y^2) - (\partial\alpha/\partial y^3) + \bigstar \\ (\partial\eta/\partial y^1) &= (\partial\gamma/\partial y^2) - (\partial\beta/\partial y^3) + \bigstar \\ (\partial\theta/\partial y^1) &= (\partial\eta/\partial y^2) - 2(\partial\zeta/\partial y^3) + \bigstar, \end{aligned} \tag{5.2}$$

where the $\bigstar$ terms involve the components $C^i{}_{jk}$, $a^i, \cdots, w^i$ in (5.1) and h_{ij}, $\partial h_{ij}/\partial y^k$, $\alpha, \cdots, \theta$ in (5.2). Since these terms result from the covariant derivatives in (1.6) and (1.12) they are of evident form.

Let us now put

$$a^i = A^i(y);\quad b^i = B^i(y);\quad c^i = C^i(y);\quad d^i = D^i(y);\quad e^i = E^i(y), \tag{5.3}$$

where the $A^i, \cdots, E^i$ are arbitrary functions of the variables y^1, y^2, y^3 each of which is analytic in the neighborhood of the values $y^1 = y^2 = y^3 = 0$; also put

$$(u^i)_{y^1=0} = P^i(y^2, y^3);\quad (v^i)_{y^1=0} = Q^i(y^2, y^3);\quad (w^i)_{y^1=0} = R^i(y^2, y^3), \tag{5.4}$$

where the P^i, Q^i, R^i are arbitrary analytic functions in the neighborhood of the values $y^2 = y^3 = 0$. Then by evaluating (5.1) at $y^i = 0$ and by differentiating (5.1) repeatedly and then evaluating at $y^i = 0$ we can calculate in a unique manner any derivative of u^i, v^i, w^i at $y^i = 0$ which involves one

or more differentiations with respect to the variable y^1.* In carrying out this process it is of course to be understood that $C^i{}_{jk} = 0$ at $y^i = 0$; also that the derivative of $C^i{}_{jk}$ with respect to y^α at $y^i = 0$, i. e. $(A^i{}_{jk\alpha})_0$, is to be expressed in terms of the quantities $a^i, \cdots, w^i$ at $y^i = 0$; and that more generally the derivative of $C^i{}_{jk}$ with respect to $y^\alpha y^{\beta_1} \cdots y^{\beta_r}$ at $y^i = 0$ which we have denoted by $(A^i{}_{jk\alpha\beta_1 \ldots \beta_r})_0$ is determined by equations of the type $M(2.10)$. The functions (5.3) and (5.4) being given the power series expansions of all functions $A^i{}_{jk\alpha}$, namely

$$(5.5) \quad A^i{}_{jk\alpha} = A^i{}_{jk\alpha}(0) + A^i{}_{jk\alpha,\beta_1}(0)y^{\beta_1} + (1/2!)A^i{}_{jk\alpha,\beta_1\beta_2}(0)y^{\beta_1}y^{\beta_2} + \cdots,$$

is uniquely determined. As so determined the number of arbitrary quantities $A^i{}_{jk\alpha}(0)$ is $24\,[= N(3,\ 1)]$ and the number of arbitrary quantities $A^i{}_{jk\alpha,\beta_1 \ldots \beta_r}(0)$ is $15K(3, r) + 9K(2, r)$; this latter number is a consequence of the 15 arbitrary functions $A^i, \cdots, E^i$ and the 9 arbitrary functions P^i, Q^i, R^i. But

$$N(3, 1 + r) = A_0 K(3, r) + A_1 K(2, r)$$

from (4.12); and since $A_0 = 15$, $A_1 = 9$ by (4.14) the above number of arbitrary quantities $A^i{}_{jk\alpha,\beta_1 \ldots \beta_r}(0)$ is equal to $N(3, 1 + r)$. Hence the equations (1.5) can not give additional conditions on the $A^i{}_{jk\alpha,\beta_1 \ldots \beta_r}(0)$ over those which result from (5.1). The constant terms $A^i{}_{jk\alpha}(0)$ and the coefficients $A^i{}_{jk\alpha,\beta_1 \ldots \beta_r}(0)$ in the series (5.5) therefore satisfy all conditions required for the functions $A^i{}_{jk\alpha}$ to be the components of a normal tensor in normal co-ordinates. As the series (5.5) converge † we can state the following theorem.

THEOREM D. *Let $A^i(y)$, $B^i(y)$, $C^i(y)$, $D^i(y)$, $E^i(y)$ denote 15 functions of the three variables y^1, y^2, y^3 and $P^i(y^2, y^3)$, $Q^i(y^2, y^3)$, $R^i(y^2, y^3)$ denote 9 functions of the two variables y^2, y^3 each function $A^i, \cdots, R^i$ being analytic in the neighborhood of the values $y^1 = y^2 = y^3 = 0$ of its arguments. Then there exists one, and only one, affine connection with components $C^i{}_{jk}$ $(= C^i{}_{kj})$ in a system of normal co-ordinates, each function $C^i{}_{jk}$ being analytic in the neighborhood of the values $y^i = 0$, such that the components $a^i, \cdots, w^i$ of the resulting normal tensor $A^i{}_{jk\alpha}(y)$ are*

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* An illustration of the method of calculating these derivatives is given by E. Goursat, "Leçons sur l'Integration des équations aux dérivées partielles du premier ordre," (1921), p. 2. The system (5.1) reduces to the type considered by Goursat when the substitution (5.3) is made and the ★ terms are neglected.

† Invariantive systems of partial differential equations (see footnote on p. 227).

$$a^i = A^i(y)\,; \quad b^i = B^i(y)\,; \quad c^i = C^i(y)\,; \quad d^i = D^i(y)$$
$$e^i = E^i(y)\,; \quad u^i = U^i(y)\,; \quad v^i = V^i(y)\,; \quad w^i = W^i(y),$$

where the U^i, V^i, W^i *are analytic functions of the variables* y^1, y^2, y^3 *in the neighborhood of the values* $y^1 = y^2 = y^3 = 0$ *such that* $U^i = P^i(y^2, y^3)$, $V^i = Q^i(y^2, y^3)$, *and* $W^i = R^i(y^2, y^3)$ *for* $y^1 = 0$.

An analogous theorem can be proved for the functions $g_{\alpha\beta,\gamma\delta}$. In brief we consider that α, β, γ are arbitrary analytic functions of the three variables y^1, y^2, y^3 and that ζ, η, θ reduce to arbitrary analytic functions of the two variables y^1, y^2 for $y^1 = 0$. We assign initial values $h_{\alpha\beta} = (g_{\alpha\beta})_0 = (g_{\beta\alpha})_0$ subject to the condition that the determinant $|\,(g_{\alpha\beta})_0\,| \neq 0$, and we impose the condition $\partial h_{\alpha\beta}/\partial y^\gamma = 0$ at the point $y^i = 0$. We note that the derivative of $\partial h_{\alpha\beta}/\partial y^\gamma$ with respect to y^δ at the point $y^i = 0$, i. e. $g_{\alpha\beta,\gamma\delta}(0)$, is expressible in terms of the quantities $\alpha, \cdots, \theta$ at $y^i = 0$; also that the derivative of $\partial h_{\alpha\beta}/\partial y^\gamma$ with respect to $y^\delta, y^{\epsilon_1} \cdots y^{\epsilon_r}$ at the point $y^i = 0$, i. e., $g_{\alpha\beta,\gamma\delta\epsilon_1 \ldots \epsilon_r}(0)$, is determined by equations of the type $M(2.16)$. As in the case above treated we can determine in a unique manner the successive coefficients of the power series expansions of the functions $g_{\alpha\beta,\gamma\delta}(y)$, namely

$$(5.6) \qquad g_{\alpha\beta,\gamma\delta} = g_{\alpha\beta,\gamma\delta}(0) + g_{\alpha\beta,\gamma\delta,\epsilon_1} y^{\epsilon_1} + (1/2!)\, g_{\alpha\beta,\gamma\delta,\epsilon_1\epsilon_2}(0) y^{\epsilon_1} y^{\epsilon_2} + \cdots.$$

Since the number of arbitrary quantities $g_{\alpha\beta,\gamma\delta,\epsilon_1 \ldots \epsilon_r}$ as above determined is $3K(3, r) + 3K(2, r)$ or $N(3, 2 + r)$ by (4.13) and (4.15) the equations (1.10) can not give additional conditions on them. The formal series (5.6) therefore satisfies all conditions required for the functions $g_{\alpha\beta,\gamma\delta}$ to be the components of a metric tensor in normal co-ordinates. Since the series (5.6) converge * we can therefore state the following theorem.

THEOREM E. *Let* $a(y)$, $b(y)$, $c(y)$ *denote three functions of the three variables* y^1, y^2, y^3 *and let* $p(y^2, y^3)$, $q(y^2, y^3)$, $r(y^2, y^3)$ *denote three functions of the two variables* y^2, y^3 *each function* $a, \cdots, r$ *being analytic in the neighborhood of the values* $y^1 = y^2 = y^3 = 0$ *of its arguments; also let* $(g_{\alpha\beta})_0 = (g_{\beta\alpha})_0$, *such that the determinant* $|\,(g_{\alpha\beta})_0\,| \neq 0$, *denote a set of six arbitrary constants. Then there exists one, and only one, fundamental tensor with components* $h_{\alpha\beta}(y) = h_{\beta\alpha}(y)$ *in a system of normal co-ordinates, each function* $h_{\alpha\beta}(y)$ *being analytic in the neighborhood of the values* $y^1 = y^2 = y^3$

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* Invariantive systems of partial differential equations (see footnote on p. 227).

$= 0$, *such that* $h_{\alpha\beta}(0) = (g_{\alpha\beta})_0$, *and such that the components* $\alpha, \cdots, \theta$ *of the resulting metric tensor* $g_{\alpha\beta,\gamma\delta}(y)$ *are*

$$\alpha = a(y)\,;\ \beta = b(y)\,;\ \gamma = c(y)\,;\quad \zeta = f(y)\,;\ \eta = g(y)\,;\ \theta = h(y),$$

where f, g, h *are analytic functions of the variables* y^1, y^2, y^3 *in the neighborhood of the values* $y^1 = y^2 = y^3 = 0$, *such that* $f = p(y^2, y^3)$, $g = q(y^2, y^3)$, *and* $h = r(y^2, y^3)$ *for* $y^1 = 0$.

In place of the arbitrary components $a^i, \cdots, e^i$ and α, β, γ in Theorems D and E certain other components can evidently be chosen as those which are completely arbitrary. It can in fact be arranged so that one of the partially arbitrary components will reduce to an arbitrary analytic function of the two variables y^2, y^3 for $y_1 = 0$, another to an arbitrary analytic function of the two variables y^1, y^3 for $y^2 = 0$, etc. From equations (5. 1) and (5. 2) all such possibilities can easily be deduced.

6. *General Case of* $n(\geqq 3)$ *Variables.* Equations (4. 12) suggest that it is possible to divide the $N(n, 1)$ independent components $A^i{}_{jka}$ into $n-1$ groups, namely a group G_0 comprising A_0 components each of which can be taken as a completely arbitrary function of the n variables y^i, a group G_1 comprising A_1 components each of which can be taken to reduce to an arbitrary function of the $n-1$ variables $y^2, \cdots, y^n$ for $y^1 = 0, \cdots$, and a group G_{n-2} comprising A_{n-2} components each of which can be taken to reduce to an arbitrary function of the variables y^{n-1}, y^n for $y^1 = \cdots = y^{n-2} = 0$. Such a division of the components $A^i{}_{jka}$ into groups G_m must comprise all of the $N(n, 1)$ independent components $A^i{}_{jka}$; this is shown by (4. 10). Similar remarks apply to the components $g_{\alpha\beta,\gamma\delta}$ on account of equations (4. 11) and (4. 13).

Let us first consider the $A^i{}_{jka}$. It is evident that the group G_0 can not be filled by selecting components $A^i{}_{jka}$ from the $N(n, 1)$ independent components $A^i{}_{jka}$ at random since they are conditioned by equations (1. 6). Our problem is to show that it is possible to solve equations (1. 6) for first derivatives of the independent components $A^i{}_{jka}$ such that the quantities for which we have solved will fall into $n-2$ mutually exclusive groups; first a group consisting of first derivatives of A_1 of the independent components $A^i{}_{jka}$ with respect to y^1 (these A_1 components $A^i{}_{jka}$ will form the group G_1), second a group consisting of first derivatives of A_2 of the independent components $A^i{}_{jka}$ with respect to y^1 and y^2 (these A_2 components $A^i{}_{jka}$ will form the group G_2), $\cdots$, and finally a group consisting of first derivatives of

A_{n-2} of the independent components $A^i{}_{jka}$ with respect to y^1, y^2, $\cdots$ and y^{n-2} (these A_{n-2} components $A^i{}_{jka}$ forming the group G_{n-2}). It is possible to state a rule for the determination of the independent components $A^i{}_{jka}$ in the groups G_m $(m = 0, \cdots, n-2)$ as follows.

Rule. The group G_m $(m = 0, \cdots, n-2)$ *for the components* $A^i{}_{jka}$ *is composed of all components that can be formed from* $A^i{}_{jka}$ *by taking* $a = m+1$; $i = 1, \cdots, n$; $j, k = 1, \cdots, n$ *subject to the inequalities* $j \leqq k$ *and* $k > m+1$. It can readily be verified that the application of this Rule gives a correct division of the components $A^i{}_{jka}$ into groups G_m. It is to be observed that the components $A^i{}_{jka}$ in a particular group G_m as given by the above Rule are independent among themselves, i. e. are not related by an equation of the form (a) and are also independent of the components $A^i{}_{jka}$ in the other groups G_m. The number of components $A^i{}_{jka}$ in the group G_m $(m = 0, \cdots, n-2)$ as determined by the above Rule is $nK(n, 2) - nK(m+1, 2)$ or A_m by (4. 14). Now let A_{lm}, where $l = 1, \cdots, A_m$; $m = 0, \cdots, n-2$, denote the A_m components $A^i{}_{jka}$ in the group G_m. On the basis of the above Rule it must therefore be possible to write equations (1. 6) in the following form in which the arrangement into groups G_m $(m = 1, \cdots, n-2)$ is also indicated:

$$
\left\{\begin{array}{ll}
\partial A_{l1}/\partial y^1 = \sum (\partial A_{pq}/\partial y^r) + \bigstar & (l = 1, \cdots, A_1) \\
\partial A_{l2}/\partial y^1 = \sum (\partial A_{pq}/\partial y^r) + \bigstar & (l = 1, \cdots, A_2) \\
\partial A_{l2}/\partial y^2 = \sum (\partial A_{pq}/\partial y^r) + \bigstar & (l = 1, \cdots, A_2) \\
\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots & \\
\partial A_{ln-2}/\partial y^1 = \sum (\partial A_{pq}/\partial y^r) + \bigstar & (l = 1, \cdots, A_{n-2}) \\
\vdots \qquad\qquad \vdots & \qquad\qquad \vdots \\
\partial A_{ln-2}/\partial y^{n-2} = \sum (\partial A_{pq}/\partial y^r) + \bigstar & (l = 1, \cdots, A_{n-2}).
\end{array}\right. \tag{6.1}
$$

The summations Σ in the right members of the above equations denote linear homogeneous expressions with constant coefficients in the derivatives $\partial A_{pq}/\partial y^r$. No derivative $\partial A_{pq}/\partial y^r$ which appears in the right member occurs also in the left member of any of these equations. The $\bigstar$ terms denote polynomials in the components A_{lm} and $C^i{}_{jk}$ such as result from the covariant derivatives in (1. 6).

It is now necessary to show that equations (1. 6) can be put into the form (6. 1). For this purpose let us first write (1. 6) as

$$(6.2)\qquad \frac{\partial A^i_{jk\alpha}}{\partial y^\beta} = \frac{\partial A^i_{jk\beta}}{\partial y^\alpha} + \frac{2}{3}\frac{\partial A^i_{j\beta\alpha}}{\partial y^k} + \frac{2}{3}\frac{\partial A^i_{k\beta\alpha}}{\partial y^j} + \frac{1}{3}\frac{\partial A^i_{\alpha\beta j}}{\partial y^k} + \frac{1}{3}\frac{\partial A^i_{\alpha\beta k}}{\partial y^j} + \bigstar.$$

Then choose the indices $ijk\alpha\beta$ so that the derivatives in the left member of (6.2) will be equal successively to the derivatives in the left members of (6.1). We thus arrive at a set of equations

$$(6.3)\qquad \partial A_{lm}/\partial y^k = \sum (\partial A_{uv}/\partial y^w) + \bigstar.$$
$$(l = 1, \cdots, A_m,\ m = 1, \cdots, n-2,\ k = 1, \cdots, m).$$

For definiteness let us now consider an equation (6.2) for which $\alpha = 2$ and $\beta = 1$, i. e.

$$(6.4)\qquad \frac{\partial A^i_{jk2}}{\partial y^1} = \frac{\partial A^i_{jk1}}{\partial y^2} + \frac{2}{3}\frac{\partial A^i_{j12}}{\partial y^k} + \frac{2}{3}\frac{\partial A^i_{k12}}{\partial y^j} + \frac{1}{3}\frac{\partial A^i_{21j}}{\partial y^k} + \frac{1}{3}\frac{\partial A^i_{21k}}{\partial y^j} + \bigstar,$$

where it is to be supposed that $j \leqq k$ and $k \geqq 3$. The component A^i_{jk2} whose derivative stands in the left member of (6.4) therefore belongs to the group G_1. With regard to the right member of (6.4) we see that A^i_{jk1} must belong to group G_0. If $j = 1$ the component A^i_{j12} belongs to group G_0 since it is equal to $-2A^i_{121}$ by (a); similarly if $j = 2$ the component A^i_{j12} belongs to group G_0. If $j \geqq 3$ the component A^i_{j12} belongs to group G_1. Since $k \geqq 3$ the next component A^i_{k12} belongs to group G_1. For $j = 1, 2$ the component A^i_{21j} belongs to group G_0 and for $j \geqq 3$ this component is equal to $-(A^i_{1j2} + A^i_{2j1})$ of which A^i_{1j2} belongs to group G_1 and A^i_{2j1} belongs to group G_0. Finally A^i_{21k} is equal to $-(A^i_{1k2} + A^i_{2k1})$ and since $k \geqq 3$ the component A^i_{1k2} belongs to group G_1 and the component A^i_{2k1} belongs to group G_0. As a consequence of the above it follows that the first set of equations (6.3), i. e.

$$\partial A_{l1}A_l/\partial y^1 = \sum \partial A_{uv}/\partial y^w + \bigstar \qquad (l = 1, \cdots, A_1),$$

contains in its right member derivatives of components A_{uv} which belong to groups G_0 and G_1 alone. Now consider equations (6.2) such that $\beta \leqq \mu$, $\alpha = \mu + 1$, $j \leqq k$ and $k > \mu + 1$. The component $A^i_{jk\alpha}$ whose derivative forms the left member of (6.2) therefore belongs to group G_μ. Let us now examine the components $A^i_{jk\alpha}$ whose derivatives are in the right member of (6.2) bearing in mind that $\alpha = \mu + 1$ and that j, k, β satisfy the above inequalities. The first component $A^i_{jk\beta}$ belongs to group $G_{\beta-1}$. The com-

ponent $A^i_{j\beta a}$ belongs to group G_μ if $j > \mu + 1$. If $j = \mu + 1$ it belongs to group $G_{\beta-1}$. Lastly if $j < \mu + 1$ the component $A^i_{j\beta a}$ is to be put equal to $-(A^i_{\beta aj} + A^i_{aj\beta})$ of which $A^i_{\beta aj}$ belongs to group G_{j-1} and $A^i_{aj\beta}$ belongs to group $G_{\beta-1}$. Since $k \geqq \mu + 2$ the component $A^i_{k\beta a}$ belongs to group G_μ. The component $A_{a\beta j}$ belongs to group G_{j-1} if $j \leqq \mu$ and to group $G_{\beta-1}$ if $j = \mu + 1$. If $j > \mu + 1$ the component $A^i_{a\beta j}$ is to be put equal to $-(A^i_{\beta ja} + A^i_{ja\beta})$ where $A^i_{\beta ja}$ belongs to group G_μ and $A^i_{ja\beta}$ to group $G_{\beta-1}$. The last component $A^i_{a\beta k}$ is to be put equal to $-(A^i_{\beta ka} + A^i_{ka\beta})$ where it is seen that $A^i_{\beta ka}$ belongs to group G_μ and that $A^i_{ka\beta}$ belongs to group $G_{\beta-1}$. This examination shows that the right members of (6. 2) in which the indices j, k, α, β are subject to the above conditions contain derivatives of components A^i_{jka} belonging to groups G_0 to G_μ inclusive. Hence in the right members of the μ-th set of equations (6. 3), i. e. the set for which $m = \mu$, there can occur only derivatives of components A^i_{jka} which belongs to groups G_0 to G_μ inclusive.

It can be verified directly from (6. 2) that the derivative in the left member of any equation (6. 3) can occur in the right member of the same equation with a coefficient which is not equal to 1, so that the equation can be solved for the derivative in question. We can therefore consider that equations (6. 3) are such that the derivative in the left member of any equation does not occur in the right member of the same equation. Let us now suppose that a derivative $\partial A_{lm}/\partial y^k$ occurring in the left member of one of the equations (6. 3), say the pth equation, occurs also in the right member of the qth equation of (6. 3). It will be assumed that the pth equation corresponds to a value of the index m equal to M_1 and that the qth equation corresponds to a value of m equal to M_2, such that $M_1 < M_2$. Now multiply the pth equation by the coefficient of the above derivative $\partial A_{lm}/\partial y^k$ in the qth equation and combine with the qth equation by adding corresponding members of these two equations. We thus obtain an equation from which the above derivative $\partial A_{lm}/\partial y^k$ can be cancelled. Moreover the equation so obtained will contain as its left member the derivative in the left member of the qth equation and this derivative can not be cancelled out on account of the above result, namely that the right members of the set of equations (6. 3) for which $m = \mu$ can contain only derivatives of components A^i_{jka} belonging to groups G_0 to G_μ inclusive. The equation obtained by combination of the pth and qth equations will therefore be considered to replace the qth equation in (6. 3). Now suppose that the pth and qth equations correspond to the same value of the index m. Taking this value of m to be μ let us then consider equations (6. 2) for $\alpha = \mu + 1$, $\beta \leqq \mu$, $j \leqq k$, $k > \mu + 1$ which

determine the μth set of equations (6.3). From the derivatives $\partial A^i{}_{k\beta a}/\partial y^j$ and $\partial A^i{}_{a\beta k}/\partial y^j$ for $j \leqq \mu$ in the right members of (6.2), and from these derivatives alone, can we obtain derivatives occurring in the left members of the μth set of equations (6.3). Since $A^i{}_{a\beta k} = -A^i{}_{\beta ka} - A^i{}_{ka\beta}$ where $A^i{}_{\beta ka}$ belongs to group G_μ and $A^i{}_{ka\beta}$ belongs to group G_ν where $\nu < \mu$, only the derivatives $\partial A^i{}_{k\beta a}/\partial y^j$ for $j < \mu$ need be considered. We can therefore write

$$p\text{th equation:} \qquad \frac{\partial A^i{}_{jka}}{\partial y^\beta} = \frac{1}{3}\frac{\partial A^i{}_{\beta ka}}{\partial y^j} + \cdots$$

$$q\text{th equation:} \qquad \frac{\partial A^i{}_{\beta ka}}{\partial y^j} = \frac{1}{3}\frac{\partial A^i{}_{jka}}{\partial y^\beta} + \cdots,$$

where the dots denote quantities which do not enter into consideration. When the derivative $\partial A^i{}_{jka}/\partial y^\beta$ in the right member of the qth equation is eliminated by means of the pth equation we obtain an equation which can be solved for the derivative $\partial A^i{}_{\beta ka}/\partial y^j$. This latter equation will replace the qth equation in the set (6.3). By continuing this process of combining equations (6.3) it is obviously possible to obtain equations (6.1). For the special case $n = 3$ the components $A^i{}_{jka}$ in groups G_0 and G_1 as determined by the above Rule are given in the tables in § 5; also equations (6.1) assume the particular form (5.1).

The operations performed on equations (6.3) in order to deduce the μth set of equations (6.1) involved only those equations (6.3) for which $m \leqq \mu$. Hence the above result concerning the groups of the components $A^i{}_{jka}$ whose derivatives occur in the right members of (6.3) applies to equations (6.1). This result is stated as the following lemma.

LEMMA I. *For any derivative $\partial A_{pq}/\partial y^r$ in the right members of the set of equations*

$$\partial A_{lm}/\partial y^k = \sum \partial A_{pq}/\partial y^r + \bigstar$$

$$(m = \mu, \quad 1 \leqq \mu \leqq n-2, \quad l = 1, \cdots, A_\mu, \quad k = 1, \cdots, \mu)$$

selected from (6.1) *the inequalities $q \leqq \mu$, $r > q$ are satisfied.* Denoting the derivative in the left member of any equation (6.1) by $\partial A_{lm}/\partial y^k$ so that $k \leqq m$ we can show that

$$(6.5) \qquad \frac{\partial^{s+1} A_{lm}}{\partial y^k \partial y^{\alpha_1} \cdots \partial y^{\alpha_s}} = \sum \frac{\partial^{s+1} A_{pq}}{\partial y^r \partial y^{\sigma_1} \cdots \partial y^{\sigma_s}} + \sum \frac{\partial^s \bigstar}{\partial y^{\alpha_1} \cdots \partial y^{\alpha_s}},$$

where $s = 1, 2, \cdots$; $\alpha_1, \cdots, \alpha_s = 1, \cdots, n$; $r, \sigma_1, \cdots, \sigma_s > q$, i.e. each of the indices $r, \sigma_1, \cdots, \sigma_s$ is greater than q. Each of the summations $\sum$ represents a linear homogeneous expression with constant coefficients in the

derivatives of the type indicated. It is evident that (6.5) holds for $\alpha_1, \cdots, \alpha_s = n-1, n$ since (6.5) then results directly by differentiation of (6.1). Now differentiate any equation (6.5) for $\alpha_1, \cdots, \alpha_s = n-1, n$ with respect to y^{n-2}. Then all derivatives in the right member of the resulting equation will be of the type which appears in the right member of (6.5) except derivatives of A_{pn-2}. These latter derivatives, however, can be eliminated by a substitution (6.5) for $\alpha_1, \cdots, \alpha_s = n-1, n$; hence (6.5) is true for $\alpha_1 = n-2$; $\alpha_2, \cdots, \alpha_s = n-1, n$. Differentiating any equation (6.5) for $\alpha_1 = n-2$; $\alpha_2, \cdots, \alpha_s = n-1, n$ with respect to y^{n-2} we find in a similar manner that (6.5) is true for $\alpha_1, \alpha_2 = n-2$; $\alpha_3, \cdots, \alpha_s = n-1, n$; etc. Hence (6.5) holds for $\alpha_1, \cdots, \alpha_s = n-2, n-1, n$. If we now differentiate any equation (6.5) for $\alpha_1, \cdots, \alpha_s = n-2, n-1, n$ with respect to y^{n-3} and eliminate the derivatives of A_{pn-3}, A_{pn-2} which occur, we obtain (6.5) for $\alpha_1 = n-3$; $\alpha_2, \cdots, \alpha_s = n-2, n-1, n$. Then differentiating (6.5) for $\alpha_1 = n-3$; $\alpha_2, \cdots, \alpha_s = n-2, n-1, n$, we obtain (6.5) for $\alpha_1, \alpha_2 = n-3$; $\alpha_3, \cdots, \alpha_s = n-2, n-1, n$, etc. Hence (6.5) is true for $\alpha_1, \cdots, \alpha_s = n-3, n-2, n-1, n$. Continuing this process we finally obtain (6.5) for $\alpha_1, \cdots, \alpha_s = 1, \cdots, n$, as was to be proved.

We next denote by ϕ_{lm}, where $l = 1, \cdots, A_m$; $m = 0, 1, \cdots, n-2$, an arbitrary function of the variables $y^{m+1}, y^{m+2}, \cdots, y^n$ analytic in the neighborhood of the values $y^{m+1} = \cdots = y^n = 0$. Now put $A_{l0} = \phi_{l0}$ $(l = 1, \cdots, A_0)$; also put $A_{lm} = \phi_{lm}$ $(l = 1, \cdots, A_m;\ m = 1, \cdots, n-2)$ for $y^1 = \cdots = y^m = 0$. From the functions ϕ_{lm} and equations (6.5) we can then calculate the successive coefficients $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}(0)$ in the power series expansions of the functions $A^i{}_{jk\alpha}$ about the point $y^i = 0$, i.e. we can determine the series

$$(6.6)\qquad A^i{}_{jk\alpha} = A^i{}_{jk\alpha}(0) + A^i{}_{jk\alpha,\beta_1}(0)y^{\beta_1} + (1/2!)A^i{}_{jk\alpha,\beta_1\beta_2}y^{\beta_1}y^{\beta_2} + \cdots,$$

where the constant terms $A^i{}_{jk\alpha}(0)$ are given by the functions ϕ_{lm} evaluated at $y^i = 0$. In the determination of the quantities $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}(0)$ the initial conditions $C^i{}_{jk} = 0$ at $y^i = 0$ are to be imposed; also $\partial C^i{}_{jk}/\partial y^\alpha$ at $y^i = 0$ is equal to $A^i{}_{jk\alpha}(0)$; and the higher derivatives of $C^i{}_{jk}$ at $y^i = 0$ are determined by equations of the type $M(2.10)$. The number of arbitrary quantities $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}(0)$ is $A_0K(n, r) + A_1K(n-1, r) + \cdots + A_{n-2}K(2, 4)$ which is equal to $N(n, 1+r)$ by (4.12). These $N(n, 1+r)$ arbitrary quantities $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}(0)$ being given, the remainder are determined by the above calculation. The determination of the quantities $A^i{}_{jk\alpha,\beta_1\ldots\beta_r}(0)$ will be unique, for if this were not the case we would be led to equations as a result of

conditions of integrability which would reduce the number of arbitrary quantities $A^i{}_{jka,\beta_1\ldots\beta_r}(0)$ to a number less than $N(n, 1+r)$. Equations (1.5) cannot therefore give additional conditions on the quantities $A^i{}_{jka,\beta_1\ldots\beta_r}(0)$ so that the series (6.6) if convergent will define a set of functions $A^i{}_{jka}(y)$ which will be the components of a normal tensor in a system of normal coordinates. Since it can be shown that the formal power series (6.6) converge * it follows that by Theorem A we therefore have the following theorem.

THEOREM F. *Let $\phi_{lm}(y)$, where $l=1,\cdots,A_m$; $m=0, 1,\cdots,n-2$, denote an arbitrary function of the variables y^{m+1}, $y^{m+2},\cdots,y^n$ which is analytic in the neighborhood of the values $y^{m+1}=\cdots=y^n=0$; also let A_{lm}, where $l=1,\cdots,A_m$, denote the components $A^i{}_{jka}$ in the group G_m $(m=0, 1,\cdots,n-2)$. Then there exists one, and only one, affine connection with components $C^i{}_{jk}(y)=C^i{}_{kj}(y)$ in a system of normal coordinates y^i, each function $C^i{}_{jk}$ being analytic in the neighborhood of the values $y^i=0$, such that the components A_{lm} of the resulting normal tensor are*

$$A_{l0}=\phi_{l0} \qquad (l=1,\cdots,A_0)\,;\ \textit{and}$$
$$A_{lm}=\phi_{lm} \qquad (l=1,\cdots,A_m;\ m=1,\cdots,n-2)$$

for $y^1=\cdots=y^m=0$.

As the discussion of the functions $g_{\alpha\beta,\gamma\delta}$ is to be made in an analogous manner the special details of the argument can be omitted. The separation of the $g_{\alpha\beta,\gamma\delta}$ into groups G_m $(m=0,\cdots,n-2)$ made on the basis of the equations

$$\frac{\partial g_{\alpha\beta,\gamma\delta}}{\partial y^\epsilon}=\frac{1}{2}\left[\frac{\partial g_{\epsilon\beta,\gamma\delta}}{\partial y^\alpha}+\frac{\partial g_{\alpha\epsilon,\gamma\delta}}{\partial y^\beta}+\frac{\partial g_{\alpha\beta,\epsilon\delta}}{\partial y^\gamma}+\frac{\partial g_{\alpha\beta,\gamma\epsilon}}{\partial y^\delta}\right]+\bigstar, \tag{6.7}$$

which are equivalent to (1.12), is given by the following rule.

Rule. The group G_m $(m=0,\cdots,n-2)$ for the components $g_{\alpha\beta,\gamma\delta}$ is composed of all components that can be formed from $g_{\alpha\beta,\gamma\delta}$ by taking $\alpha=m+1$; $\beta,\ \gamma,\ \delta=1,\cdots,n$ subject to the inequalities $\beta\leqq m+1$, $\beta<\gamma$, $\gamma\leqq\delta$ and $\delta>m+1$.

The components $g_{\alpha\beta,\gamma\delta}$ in the above groups G_m are to be taken as independent which can be done since they are not connected by relations of the form (b). The number of components $g_{\alpha\beta,\gamma\delta}$ in the group G_m $(m=0,\cdots,n-2)$ is $(m+1)\,K(n,2)-nK(m+1,2)$ which by (4.15) is equal to B_m. Letting B_{lm}, where $l=1,\cdots,B_m$; $m=0, 1,\cdots,n-2$, denote

* Invariantive systems of partial differential equations (see footnote on p. 227).

the B_m components in the group G_m, we arrive at equations analogous to (6.1) namely

$$
\begin{aligned}
&\partial B_{l1}/\partial y^1 = \sum (\partial B_{pq}/\partial y^r) + \bigstar && (l = 1, \cdots, B_1)\\
&\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\\
&\left.\begin{array}{ll}\partial B_{ln-2}/\partial y^1 = \sum (\partial B_{pq}/\partial y^r) + \bigstar & (l = 1, \cdots, B_{n-2})\\ \vdots \qquad\qquad \vdots \qquad\quad \vdots & \\ \partial B_{ln-2}/\partial y^{n-2} = \sum (\partial B_{pq}/\partial y^r) + \bigstar & (l = 1, \cdots, B_{n-2}).\end{array}\right\}
\end{aligned}
\tag{6.8}
$$

Equations (6.8) constitute the solution of equations (6.7) for the derivatives which appear in the left members of (6.8). For the special case $n = 3$ equations (6.8) assume the particular form (5.2). For this case also the components $g_{\alpha\beta,\gamma\delta}$ in the groups G_0 and G_1 are given in the tables in § 5. The following lemma corresponds to Lemma I.

LEMMA II. *The inequalities $q \leqq \mu$, $r > q$ are satisfied for any derivative $\partial B_{pq}/\partial y^r$ in the right members of the set of equations*

$$\partial B_{lm}/\partial y^k = \sum (\partial B_{pq}/\partial y^r) + \bigstar$$

$$(m = \mu, \quad 1 \leqq \mu \leqq n-2, \quad l = 1, \cdots, B_\mu, \quad k = 1, \cdots, \mu)$$

selected from (6.8). Corresponding to (6.5) we now have the equations

$$\frac{\partial^{s+1} B_{lm}}{\partial y^k \partial y^{\alpha_1} \cdots \partial y^{\alpha_s}} = \sum \frac{\partial^{s+1} B_{pq}}{\partial y^r \partial y^{\sigma_1} \cdots \partial y^{\sigma_s}} + \sum \frac{\partial^s \bigstar}{\partial y^{\alpha_1} \cdots \partial y^{\alpha_s}}, \tag{6.9}$$

where $k \leqq m$; $s = 1, 2, \cdots$; $\alpha_1, \cdots, \alpha_s = 1, \cdots, n$; and $r, \sigma_1, \cdots, \sigma_s < q$. Denoting by ψ_{lm}, where $l = 1, \cdots, B_m$; $m = 0, 1, \cdots, n-2$, an arbitrary function of the variables $y^{m+1}, \cdots, y^n$ analytic in the neighborhood of the values $y^{m+1} = \cdots = y^n = 0$, we put $B_{l0} = \psi_{l0}$ $(l = 1, \cdots, B_0)$, and for $y^1 = \cdots = y^m = 0$ we put $B_{lm} = \psi_{lm}$ $(l = 1, \cdots, B_m;\ m = 1, \cdots, n-2)$. Now assign the initial conditions $h_{\alpha\beta} = (g_{\alpha\beta})_0 = (g_{\beta\alpha})_0$ such that the determinant $|\,(g_{\alpha\beta})_0\,| \neq 0$ and $\partial h_{\alpha\beta}/\partial y^\gamma = 0$ at $y^i = 0$; also put $\partial^2 h_{\alpha\beta}/\partial y^\gamma \partial y^\delta$ at $y^i = 0$ equal to $g_{\alpha\beta,\gamma\delta}(0)$, which is determined from the given functions ψ_{lm}, and determine the higher derivatives of $h_{\alpha\beta}$ at $y^i = 0$ by equations of the type $M(2.16)$. From the functions and equations (6.9) the successive coefficients $g_{\alpha\beta,\gamma\delta,\epsilon_1 \ldots \epsilon_r}(0)$ in the series

$$g_{\alpha\beta,\gamma\delta} = g_{\alpha\beta,\gamma\delta}(0) + g_{\alpha\beta,\gamma\delta,\epsilon_1}(0) y^{\epsilon_1} + (1/2!) g_{\alpha\beta,\gamma\delta,\epsilon_1\epsilon_2}(0) y^{\epsilon_1} y^{\epsilon_2} + \cdots \tag{6.10}$$

are then uniquely determined. The number of arbitrary quantities $g_{\alpha\beta,\gamma\delta,\epsilon_1 \ldots \epsilon_r}$ is $N(n, 2+r)$ by (4.13), and since the remaining quantities $g_{\alpha\beta,\gamma\delta,\epsilon_1 \ldots \epsilon_r}$ are determined, e. g. by (6.9), when these arbitrary ones are given, the

equations (1.10) can not furnish additional conditions on the quantities $g_{\alpha\beta,\gamma\delta,\epsilon_1 \ldots \epsilon_r}(0)$.

It can be shown that the series (6.10) converge;* hence by theorem B we have the following theorem.

THEOREM G. *Let* ψ_{lm}, *where* $l = 1, \cdots, B_m$, $m = 0, 1, \cdots, n-2$, *denote an arbitrary function of the variables* $y^{m+1}, y^{m+2}, \cdots, y^n$ *which is analytic in the neighborhood of the values* $y^{m+1} = \cdots = y^n = 0$; *let* $(g_{\alpha\beta})_0 = (g_{\beta\alpha})_0$, *where* $\alpha, \beta = 1, \cdots, n$, *be arbitrary constants such that the determinant* $|(g_{\alpha\beta})_0| \neq 0$; *also let* B_{lm}, *where* $l = 1, \cdots, B_m$, *denote the components* $g_{\alpha\beta,\gamma\delta}$ *in the group* G_m $(m = 0, 1, \cdots, n-2)$. *Then there exists one, and only one, fundamental tensor with components* $h_{\alpha\beta}(y) = h_{\beta\alpha}(y)$ *in a system of normal co-ordinates* y^i, *each function* $h_{\alpha\beta}(y)$ *being analytic in the neighborhood of the values* $y^i = 0$, *such that* $h_{\alpha\beta}(0) = (g_{\alpha\beta})_0$, *and the components* B_{lm} *of the resulting metric tensor are*

$$B_{l0} = \psi_{l0} \qquad (l = 1, \cdots, B_0); \text{ and}$$
$$B_{lm} = \psi_{lm} \qquad (l = 1, \cdots, B_m;\ m = 1, \cdots, n-2)$$

for $y^i = \cdots = y^m = 0$.

PRINCETON, N. J.

* Invariantive systems of partial differential equations (see footnote on p. 227).

Relations Between the Critical Points of a Real Analytic Functions of N Independent Variables.*

By Arthur B. Brown.†

1. *Introduction. A critical point of a function is one where all its first partial derivatives are zero.* Birkhoff was the first to state a relation between the critical points of a function of n variables.‡ Morse § went further, obtaining a complete set of relations in the important (and general) case of non-degenerate critical points; that is, where the Hessian of the function is not zero at any critical point. We shall find it convenient to borrow certain results from his paper. Morse's results are described in the second part of the present paper.

In this paper we investigate the case of the general isolated critical point, dropping the restriction used by Morse, but requiring the function to be analytic. Relations similar to Morse's are obtained. They may be described roughly by saying that each general isolated critical point has the effect of a whole group of non-degenerate critical points.

I. Preliminary Theorems of Analysis Situs.

2. *Notations and definitions.* We shall use the nomenclature of analysis situs as defined by Alexander,¶ substituting the terms *cycle* and *Betti number* for *closed chain* and *connectivity number* respectively. Notations not found in Alexander's article will be as defined in Veblen's Colloquium Lectures on Analysis Situs.

In the entire paper all geometric terms and operations such as Betti number, addition of chains, bounding, may be taken either modulo 2 or absolute. For definiteness we shall take the ordinary (absolute) case, but it is understood that the work would go through as well in the modulo 2 case.

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† National Research Fellow.

‡ For references on critical points see the author's paper "Relations between the Critical Points and Curves of a Real Analytic Function of Two Independent Variables," *Annals of Mathematics*, Vol. 31 (1930).

§ Marston Morse, "Relations between the Critical Points of a Real Function of n Independent Variables," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 345-396.

¶ J. W. Alexander, "Combinatorial Analysis Situs," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 301-329.

3. *Relations between nested complexes.*

THEOREM 1. *Let C and K be given complexes, where K contains C in the point-set sense. Let a deformation D exist which keeps points of K on K, and carries K into a complex on C. Then the Betti numbers of C are at least as great as those of K.*

Proof. Let A be a complete set of non-bounding i-cycles of C, where i is any non-negative integer. The theorem will be proved if we can show that any i-cycle of K is dependent * on cycles of A.

Let E be any i-cycle of K. Let E_1 be the transform of E by D. Then E_1 may be a singular i-cycle.

Since A is a complete set of non-bounding i-cycles of C, we know that there exists an integer $m \neq 0$ and a combination A_2 of cycles of A such that

$$mE_1 + A_2 \sim 0 \quad \text{on } C, \text{ hence also on } K.$$

Now

$$mE - mE_1 \sim 0 \quad \text{on } K,$$

as follows from the properties of a deformation. By addition of the two homologies we obtain the following.

$$mE + A_2 \sim 0 \quad \text{on } K.$$

Hence the theorem is proved.

THEOREM 2. *Let C and K be given complexes, where K contains C in the point-set sense. Let a deformation D exist which keeps points of K on K, keeps points of C on C, and carries K into a complex on C. Then C and K have the same Betti numbers.*†

Proof. Let A be a complete set of non-bounding i-cycles of C, where i is any non-negative integer. Now the hypotheses of Theorem 1 are satisfied, and in proving Theorem 1 we showed that any i-cycle of K is dependent on cycles of A. If we can show that there is no homology on K among the cycles of A, it will follow that A is a complete set of non-bounding i-cycles of K, and Theorem 2 will be proved.

Let A_1 be any linear combination of cycles of A, with integral coefficients, not all zero. Let A_2 be the transform of A_1 by D. Since the transforms of A_1 at all intermediate stages of D lie on C, the following homology is valid.

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* Dependence shall always refer to homologies.

† It may be seen from the proofs that Theorems 1 and 2 apply to any geometrical configurations for which Betti numbers, topologically defined, exist.

$$A_1 - A_2 \sim 0 \quad \text{on } C.$$

Now if A_1 bounded any complex, say E_1, of K, then A_2 would bound the transform of E_1 by D. Since that transform would be on C, we would have the following homology.

$$A_2 \sim 0 \quad \text{on } C.$$

Adding to the last homology, we would have the following.

$$A_1 \sim 0 \quad \text{on } C.$$

But that is impossible, since A is a complete set of non-bounding i-cycles of C. Hence there are no homologies on K among the cycles of A, and Theorem 2 is true.

4. *Relations between the Betti numbers of overlapping complexes.* The following theorem will be used later in the paper to obtain relations between the critical points.

THEOREM 3. *Let A and B be complexes of dimensionalities m and n respectively, $m \leqq n$. Let the points common to A and B form a complex C. Let the set of all points that belong to either A or B, or both, form a complex D which can be so divided into cells that at the same time A, B and C receive proper divisions into cells.*

Then integers α_i, $i = 0, 1, \cdots, n$; and β_j, $j = 0, 1, \cdots, n+1$; exist, all positive or zero, with $\beta_0 = \beta_{n+1} = 0$, such that the following relations hold: *

$$R_i(C) = \beta_{i+1} + \alpha_i, \tag{4.1}$$

$$R_i(D) - R_i(A) - R_i(B) = \beta_i - \alpha_i, \quad i = 0, 1, \cdots, n. \tag{4.2}$$

Proof. The proof will require a number of steps, which we shall designate as lemmas. We begin by introducing notation for certain sets of cycles.

Let A^i and B^i be maximum sets of i-cycles of A and B respectively, which are independent, on A and B respectively, of cycles of C.

Let C_a^i and C_b^i be maximum sets of i-cycles of C which bound on A and B respectively, but are independent non-bounding on B and A respectively.

Let C_1^i be a maximum set of i-cycles of C which are independent non-bounding on D.

Let C_d^i be a maximum set of independent i-cycles of C, each of which bounds both on A and on B.

---

* We let $R_i(K)$ denote the i-th Betti number of K.

The number of cycles in any set will be denoted by replacing the large letter by a small letter. Thus C_1^i contains c_1^i cycles.

LEMMA 1. $R_i(A) = a^i + c_1^i + c_b^i$.

Proof. The sets A^i, C_1^i, C_b^i are composed of cycles of A. That they contain a complete set of non-bounding i-cycles of A is easily verified from the definitions above. If we supposed a homology existed among them on A, consideration of these definitions will show successively that it could involve no cycles of A^i, none of C_1^i, and none of C_b^i. Hence there is no such homology, and the lemma is true.

LEMMA 2. $R_i(B) = b^i + c_1^i + c_a^i$.

Proof. This follows from Lemma 1 because of the symmetry between A and B.

LEMMA 3. *The cycles C_1^i, C_a^i, C_b^i, C_d^i are independent on C.*

Proof. If there were any homology among them, it could not involve any cycles of C_1^i, since the latter are independent on D, and all the others are bounding on D. Then it could involve no cycles of C_a^i, since they are independent on B. For a similar reason it could involve no cycles of C_b^i. Hence it must be a homology, on C, among the cycles of C_d^i, which is contrary to the definition of that set. Therefore there can be no such homology, and the lemma is proved.

LEMMA 4. $R_i(C) = c_1^i + c_a^i + c_b^i + c_d^i$.

Proof. In view of Lemma 3 it will be sufficient to show that if E^i is any i-cycle of C, then it is dependent on cycles of the sets mentioned in Lemma 3.

From the definition of C_1^i it follows that there exists an integer $m \neq 0$, and a (possibly empty) combination E_1^i of cycles of C_1^i such that

$$mE^i + E_1^i \sim 0 \quad \text{on } D. \tag{4.3}$$

Let P^i be the boundary of the part in A of an $(i+1)$-chain K^{i+1} corresponding to the last homology. Then

$$F^i = mE^i + E_1^i - P^i \sim 0 \quad \text{on } B. \tag{4.4}$$

That is because F^i is the boundary of

$$L^{i+1} = (K^{i+1} \text{ minus the part of } K^{i+1} \text{ in } A).$$

Now either there is a non-zero multiple of F^i that bounds in A, or there is not. In the first case F^i must be dependent, on C, on cycles of $C_a{}^i$, and in the second it must be dependent, on A, on cycles of $C_b{}^i$. That follows from the definitions of $C_a{}^i$ and $C_b{}^i$. It is therefore correct to say, regardless of which case is at hand, that an integer $q \neq 0$ and a (possibly empty) combination $E_b{}^i$ of cycles of $C_b{}^i$ and $C_d{}^i$, exist such that

$$qF^i + E_b{}^i \sim 0 \quad \text{on } A. \tag{4.5}$$

By similar proof we can show that there exists an integer $r \neq 0$ and a combination $E_a{}^i$ of cycles of $C_a{}^i$ and $C_d{}^i$, such that

$$rP^i + E_a{}^i \sim 0 \quad \text{on } B. \tag{4.6}$$

In proving this we would use, in place of (4.4), the homology

$$P^i \sim 0 \quad \text{on } A. \tag{4.7}$$

From (4.4) and (4.7) we conclude that (4.5) holds also on B, and (4.6) holds also on A. Thus each holds both on A and on B. From the definition of $C_d{}^i$ we conclude that there exist integers s and t, neither zero, and (possibly empty) combinations $H_1{}^i$ and $H_2{}^i$ of cycles of $C_d{}^i$, such that

$$sqF^i + sE_b{}^i + H_1{}^i \sim 0 \quad \text{on } C,$$
$$trP^i + tE_a{}^i + H_2{}^i \sim 0 \quad \text{on } C.$$

Let us multiply the first of these homologies by tr, the second by sq, and add. One term of the result is

$$trsq(F^i + P^i),$$

which, on comparison with (4.4), can be written as

$$trsq(mE^i + E_1{}^i).$$

The other terms are combinations of cycles of $C_a{}^i$, $C_b{}^i$ and $C_d{}^i$. It follows that E^i is dependent, on C, on cycles of $C_a{}^i$, $C_b{}^i$, $C_1{}^i$ and $C_d{}^i$; and Lemma 4 is proved.

Before stating the next lemma we shall introduce a new set of cycles. Given any cycle of $C_d{}^{i-1}$, it bounds both in A and in B, by definition of $C_d{}^{i-1}$. If it bounds $D_a{}^i$ in A and $D_b{}^i$ in B, then $D_a{}^i - D_b{}^i$ has no boundary, hence is a cycle. Thus for each cycle of $C_d{}^{i-1}$ can be found a corresponding cycle of D.

Let D^i be a set of i-cycles of D obtained in this way. We note that $c_d{}^{i-1}$ is the number of cycles in D^i.

LEMMA 5. *The cycles A^i, B^i, $C_1{}^i$, D^i are independent on D.*

Proof. Suppose we have a homology on D among the four sets of cycles. We shall show that it can involve no cycles.

Corresponding to the homology is a relation of bounding,

$$E^{i+1} \to P_a{}^i + P_b{}^i + P_1{}^i + P_d{}^i. \tag{4.8}$$

Here the last four symbols denote the combinations of cycles of A^i, B^i, $C_1{}^i$, D^i, respectively, involved.

Let $E_a{}^{i+1}$ be the part of E^{i+1} in A, and let F^i be the boundary of $E_a{}^{i+1}$. Let

$$E_b{}^{i+1} = E^{i+1} - E_a{}^{i+1},$$

and let H^i be the boundary of $E_b{}^{i+1}$.

Let K^{i-1} be the combination of cycles of $C_d{}^{i-1}$ corresponding to the combination $P_d{}^i$ of cycles of D^i. Let $L_a{}^i$ and $L_b{}^i$ be the chains of A and respectively, bounded by K^{i-1}, such that

$$P_d{}^i = L_a{}^i - L_b{}^i.$$

Now $(F^i - P_a{}^i - L_a{}^i)$ is an i-chain of C bounded by $(-K^{i-1})$, for the first two terms in the first parentheses represent cycles. Since the cycles of $C_d{}^{i-1}$ are independent on C, it follows that K^{i-1} must be a combination of them with all zero coefficients. Therefore $P_d{}^i$ contains no cycles.

Since F^i and H^i are the boundaries of $E_a{}^{i+1}$ and $E_b{}^{i+1}$ respectively, the following are correct relations:

$$E_a{}^{i+1} \to P_a{}^i + (F^i - P_a{}^i), \tag{4.9}$$

$$E_b{}^{i+1} \to P_b{}^i + (H^i - P_b{}^i). \tag{4.10}$$

Now $E_a{}^{i+1}$ is a chain of A, and $(F^i - P_a{}^i)$ is a set of cycles of C. It follows from (4.9) that $P_a{}^i$ is homologous on A to cycles of C. Since $P_a{}^i$ is a combination of cycles of A^i, such a homology is impossible unless $P_a{}^i$ is empty. Hence $P_a{}^i$ contains no cycles, and by similar proof, using (4.10), $P_b{}^i$ is empty.

Therefore (4.8) tells us that $P_1{}^i$ is bounding on D. Since the cycles of $C_1{}^i$ are independent on D, it follows that $P_1{}^i$ is empty. Hence the given homology involves a combination of the cycles of A^i, B^i, $C_1{}^i$, D^i with all coefficients zero. We conclude that Lemma 5 is true.

LEMMA 6. $R_i(D) = a^i + b^i + c_1{}^i + c_d{}^{i-1}$.

Proof. In view of Lemma 5 it will be sufficient to show that if E^i is any i-cycle of D, it must be dependent on cycles of A^i, B^i, $C_1{}^i$ and D^i.

Let $E_a{}^i$ be the part of E^i in A, and $E_b{}^i = E^i - E_a{}^i$. Let F^{i-1} be the boundary of $E_a{}^i$. Then F^{i-1} is a cycle of C, and bounds $(-E_b{}^i)$ as well.

From Lemma 4 we conclude that F^{i-1} is dependent, on C, on cycles of $C_1{}^{i-1}$, $C_a{}^{i-1}$, $C_b{}^{i-1}$ and $C_d{}^{i-1}$. Now $C_1{}^{i-1}$ cannot enter in the dependence, for if it did the combination of cycles of $C_1{}^{i-1}$ in question would be bounding on D, inasmuch as each of the four other sets bounds on D. Neither can $C_a{}^{i-1}$, since $C_b{}^{i-1}$, $C_d{}^{i-1}$ and F^{i-1} all bound on B. For a similar reason, $C_b{}^{i-1}$ can contribute nothing to the dependence. Hence a bounding relation of the following sort exists.

$$L^i \rightarrow mF^{i-1} + H^{i-1}.$$

Here L^i is a chain of C, m is a non-zero integer, and H^{i-1} is a combination of cycles of $C_d{}^{i-1}$.

Let $K_a{}^i$ and $K_b{}^i$ be chains of A and B respectively, bounded by H^{i-1}. That they exist follows from the definition of $C_d{}^i$. Then

$$(mE_a{}^i - L^i + K_a{}^i) \text{ is a cycle of } A, \text{ and}$$
$$(mE_b{}^i + L^i - K_b{}^i) \text{ is a cycle of } B.$$

According to Lemmas 1 and 2 these sums are, then, dependent on cycles of A^i, B^i, $C_1{}^i$, $C_a{}^i$, $C_b{}^i$. Since the last two sets contain cycles that bound on D, it follows that on D the two sums in parentheses are dependent on cycles A^i, B^i and $C_1{}^i$. Therefore the sum of the two parentheses, namely

$$mE^i + (K_a{}^i - K_b{}^i)$$

is dependent on cycles of A^i, B^i and $C_1{}^i$. But $K_a{}^i - K_b{}^i$ is a combination of cycles of D^i, as follows from the definitions of $K_a{}^i$, $K_b{}^i$ and D^i. We conclude that E^i is dependent on cycles of A^i, B^i, $C_1{}^i$ and D^i, as was to be proved.

We can now complete the proof of Theorem 3. Let $\beta_i = c_d{}^{i-1}$ and $\alpha_i = c_1{}^i + c_a{}^i + c_b{}^i$. Now if we evaluate the left hand members of (4.1) and (4.2) by means of Lemmas 1, 2, 4 and 6, and use the notation just introduced, we find that (4.1) and (4.2) are correct. It remains only to prove that $\beta_0 = \beta_{n+1} = 0$.

From its definition we find that $\beta_0 = 0$, since there are no cycles of order (-1). We showed in the proof of Lemma 5 that corresponding to each cycle of $C_d{}^i$ is a cycle of D^{i+1} which is non-bounding on D. Since there are

no $(n+1)$-cycles, it follows that there are no cycles in Ca^n, and therefore $ca^n = \beta_{n+1} = 0$. This completes the proof.

COROLLARY 1. $\alpha_i \leqq R_i(A) + R_i(B)$. $\beta_i \leqq R_i(D)$.

Proof. These inequalities can be verified by a glance at Lemmas 1 and 2, and the definitions of α_i and β_i.

COROLLARY 2. *Let A_i, B_i, C_i, D_i denote the i-th Betti numbers of A, B, C, D respectively. Then the following relations hold:*

$$(C_0 + D_0) \geqq (A_0 + B_0) \geqq D_0,$$
$$(C_0 + D_0) - (C_1 + D_1) \leqq (A_0 + B_0) - (A_1 + B_1) \leqq (C_0 + D_0) - D_1,$$
$$(C_0 + D_0) - (C_1 + D_1) + (C_2 + D_2) \geqq (A_0 + B_0) - (A_1 + B_1) + (A_2 + B_2) \geqq (C_0 + D_0) - (C_1 + D_1) + D_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(C_0 + D_0) - \cdots + (-1)^n (C_n + D_n) = (A_0 + B_0) - \cdots + (-1)^n (A_n + B_n).$$

Proof. The relations can be verified by substitution from (4.1) and (4.2) and using the relations $\beta_0 = \beta_{n+1} = 0$.

5. *Result of adding a zero-cell.*

LEMMA 7. *Let K be a given complex and P a point not in K, nor connected with it. If P is added to K, the only change in Betti numbers is an increase of unity in R_0.*

Proof. The lemma is obvious and is only stated for convenience in reference.

LEMMA 8. *Let P be a point of an ordinary n-space, and K a generalized complex in a finite part of the space, containing all the points of the space neighboring P except the point P itself. If P is added to K, the only change in Betti numbers (topologically defined), is a decrease of one in R_{n-1}.*

Proof. The addition does not change R_0, the number of parts. It does not change $R_1, R_2, \cdots$, or R_{n-2}, for any cycles of dimensionalities $1, 2, \cdots, n-2$ which pass through P can be deformed into cycles not passing through P; and any chains of dimensionalities $2, 3, \cdots, n-1$ bounded by such cycles, and passing through P, can likewise be deformed so as not to pass through P. Neither does the addition change R_n, since neither of the complexes in question fills out all n-space.

But it does reduce R_{n-1} by unity, as we show now. Let S be a small $(n-1)$-sphere with center at P. Then S, originally non-bounding, becomes bounding. Suppose U were an $(n-1)$-cycle independent of S, which became bounding when P is added to K. Then U could be deformed so as to be outside S, and since the bounded chain must contain a certain non-zero multiple, say m, of the complex consisting of S and its interior, it follows that U would be homologous on K to mS, a contradiction. (Each chain mentioned is supposed to have a definite orientation.) It follows that R_{n-1} is reduced by only one, and the lemma is proved.

II. Critical Points.

6. *Hypotheses.* The following is familiar terminology. A function of class $C^{(k)}$ is one continuous and possessing continuous partial derivatives of orders $1, 2, \cdots, k$. A locus is a regular m-spread in n-space if near any of its points it is given by expressing $(n-m)$ of the co-ordinates as functions, of class C', of the other m co-ordinates as independent variables. It is of class $C^{(k)}$ if the functions are of class $C^{(k)}$.

The hypotheses for part II are given in the next paragraph.

A closed region R of a real euclidean n-space is given, bounded by regular $(n-1)$-spreads of class $C^{(3)}$. A single-valued function f is given, real and analytic over R. Its inner normal derivative, at any boundary point, is negative. Its critical points are isolated.

Morse proved * that under these hypotheses it is possible to redefine f near the boundary so that it assumes a constant value on the boundary, greater than the value at any interior point. No new critical points are introduced by the change, and the new function is of class C''. Hereafter f will denote the new function.

7. *Morse's results.* The essential differences in the hypotheses used by Morse are as follows. The function f is not required to be analytic, but to be of class C''. The Hessian of f must not vanish at any critical point.

The critical points are then proved to be finite in number. They are divided into $n+1$ classes, defined as follows. Let

$$a_{ij} = f_{x_i x_j},$$

* Morse, *loc. cit.*, pp. 394-396.

evaluated at a given critical point. Then the form * $a_{ij}x_ix_j$ can be carried by a real, non-singular linear transformation into the form

$$-y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2. \tag{7.1}$$

The critical point is said to be of type k, where $k=0$ if all the signs in (7.1) are positive.

Let R_i be the i-th Betti number, modulo 2, of R, and M_i the number of critical points of type i, $i=0, 1, \cdots, n$. Then a set of n inequalities and one equation were obtained by Morse, in form identical with those given in Theorem 10 of this paper.

The same relations hold if the Betti numbers are taken absolute instead of modulo 2.

The corresponding results in this paper are stated in Theorem 10.

8. *Some properties of the critical points.* Morse showed by a short proof that the hypothesis of isolated critical points leads to the following conclusion.†

THEOREM 4. *The number of critical points is finite.*

A slight extension of a result demonstrated by Morse ‡ yields the following lemma.

LEMMA 9. *Let P be any point not a critical point, where f is analytic. The loci $f=$ constant neighboring P are regular analytic $(n-1)$-spreads.§ Their orthogonal trajectories are regular and analytic, form a field, and can be represented by the equations*

$$x_i = x_i(x_1^0, \cdots, x_n^0, t), \qquad i=1, 2, \cdots, n,$$

where $(x_1^0, \cdots, x_n^0)$ is the point of intersection of the trajectory with the locus $f=$ constant through P, the x_i are analytic functions, and $t=f$ is the parameter on the trajectory.

Hereafter we shall refer to the orthogonal trajectories to the manifolds $f=$ constant simply as orthogonal trajectories.

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* We shall use the notation in which repetition of a subscript denotes summation from 1 to n.

† Morse, *loc. cit.*, page 348.

‡ Morse, *loc. cit.*, pp. 355-358.

§ All loci mentioned are real, unless the contrary is stated.

THEOREM 5. *If a point is isolated on a locus $f = c$, then it is a point either of relative minimum or of relative maximum, of f.*

Proof. If every neighborhood of the point, say P, contained points satisfying $f < c$ and points satisfying $f > c$, we could join a point of the first set to a point of the second set by a curve near P but not passing through P. The curve would then contain a point at which $f = c$, contrary to the hypothesis that there are no such points near P. Hence the theorem is true.

Any critical point not a point of minimum or of maximum shall be called a critical point of intermediate type.

LEMMA 10. *Let the origin of co-ordinates be a critical point of intermediate type at which $f = c$. Then in a sufficiently small neighborhood of the origin the function*

$$H \equiv (x_i f_{x_j} - x_j f_{x_i})(x_i f_{x_j} - x_j f_{x_i})$$

does not vanish at any real point of the locus $f = c$ other than the origin itself.

Geometrically, this lemma states that the normal to the locus $f = c$ does not pass through the origin.

Proof of Lemma 10.*

The points in question are the real points at which the following equations are satisfied:

$$x_i f_{x_j} - x_j f_{x_i} = 0, \qquad i, j = 1, 2, \cdots, n; \tag{8.1}$$

$$f - c = 0. \tag{8.2}$$

The points, real and complex, not the origin, satisfying these relations and neighboring the origin, lie on a finite number of configurations g of the following nature.† A configuration of degree s, $0 < s < n$, is defined when $n - s$ of the variables are expressed as single-valued functions of the other variables, say $x_1, x_2, \cdots, x_s$, on a configuration defined by the vanishing of a pseudopolynomial for which $x_1, x_2, \cdots, x_s$ are the independent variables. These functions are analytic at points where the discriminant of the pseudopolynomial is not zero (ordinary points).

Each pseudopolynomial may be taken irreducible. Let us consider one

* The outline of this proof is due to Morse.

† Osgood, *Lehrbuch der Funktionentheorie*, Vol. II, Zweite Auflage, page 132. We shall refer to this volume as Osgood II. We shall refer to Vol. I of the *Lehrbuch*, Fünfte Auflage, as Osgood I.

of them that is irreducible. Its discriminant cannot vanish identically.* Hence the points at which it does vanish lie on sub-configurations of degrees not exceeding $s-1$, defined by the vanishing of the discriminant, a function of $x_1, \cdots, x_s$. Therefore if the configuration does not contain any real ordinary points, all its real points are confined to the configurations of lower dimensionalities which result when we set the discriminant equal to zero.

It will, then, be sufficient to consider configurations containing real ordinary points. We shall show that on each of these the function

$$r^2 = x_i x_i$$

is constant. Now if a configuration, say g_1, is of degree s_1, then in the neighborhood of any ordinary point on it, r^2 can be expressed as an analytic function of s_1 of the variables $x_1, x_2, \cdots, x_n$ as independent variables. It is possible to continue analytically from any ordinary point to any second ordinary point of g_1.† Therefore, if we can show that r^2 is constant in the neighborhood of a single ordinary point of g_1, it must equal the same constant at all ordinary points. Since r^2 is a continuous function, and there are ordinary points neighboring any point not itself an ordinary point, it will then follow that r^2 equals the same constant over the entire configuration g_1.

Let P be a real ordinary point (not the origin) on g_1. Let Q be any nearby point of g_1. Let t denote an arbitrary one of the independent variables for the configuration g_1. Since (8. 2) is satisfied, we can differentiate at Q, to obtain the following relation:

$$f_{x_i} \partial x_i / \partial t = 0. \tag{8.3}$$

Since P is a real point not the origin, and Q is near P, it follows that Q is neither a critical point nor the origin. Hence the functions

$$x_j x_j \quad \text{and} \quad f_{x_j} f_{x_j}$$

are not zero at Q. Therefore, since equations (8. 1) are satisfied on g_1, the following relations hold at Q:

$$f_{x_i} = \pm \left[(f_{x_j} f_{x_j})^{\frac{1}{2}} / (x_j x_j)^{\frac{1}{2}} \right] \cdot x_i. \tag{8.4}$$

Here the sign is the same for all n values of i.

Let us substitute now from (8. 4) in (8. 3), to obtain the following relation:

$$\left[(f_{x_j} f_{x_j})^{\frac{1}{2}} / (x_j x_j)^{\frac{1}{2}} \right] \cdot \left[x_i \cdot (\partial x_i / \partial t) \right] = 0. \tag{8.5}$$

* Osgood II, page 108, 3 Satz.

† Osgood II, page 109.

Since the first factor is not zero, we conclude that the last factor is zero. Hence

$$\partial(x_i x_i)/\partial t = 0$$

at Q. Therefore all the first partial derivatives of r^2 with respect to the independent variables are zero at all points Q neighboring P. Consequently r^2 is constant in the neighborhood of P, on g_1. Therefore r^2 has the same value over the entire configuration g_1.

Hence r^2 has only a finite number of values for all real points satisfying (8.1) and (8.2) simultaneously. Therefore any sufficiently small neighborhood of the origin contains none of these points. The proof of Lemma 10 is now complete.

9. *The trajectories* τ. We shall obtain a set of regular analytic trajectories which form a field in the neighborhood of any real point at which

$$H \equiv (x_i f_{xj} - x_j f_{x_i})(x_i f_{xj} - x_j f_{x_i}) \neq 0. \tag{9.1}$$

Let us consider the following system of differential equations,

$$dx_i/dt = \rho[x_i(f_{xj} f_{xj}) - f_{x_i}(x_j f_{xj})], \tag{9.2}$$

where

$$\rho = [2(x_i x_i)^{\frac{1}{2}}/H(x_1, \cdots, x_n)]. \tag{9.3}$$

The brackets in (9.2) were obtained as direction components of the intersection of the tangent $(n-1)$-plane to the locus $f =$ constant at a point P, with the 2-plane passing through the origin and the normal at P to the tangent $(n-1)$-plane.

LEMMA 11. *In the neighborhood of any point at which* (9.1) *is satisfied, there exists a set of regular analytic trajectories* τ, *defined by* (9.2) *and* (9.3), *which form a field. The function* f *is constant along any one of these trajectories, and the parameter* t *can be taken equal to* $r = (x_i x_i)^{\frac{1}{2}}$.

Proof. The right hand members of (9.2) are analytic. If the right hand members of any two of these equations are zero, then we can show very easily that a corresponding term of the sum in (9.1) is zero. Since $H \neq 0$, it follows that the right hand members of (9.2) are not all zero. Hence a regular analytic field of trajectories is defined.

Along any one of these trajectories,

$$df/dt = f_{x_i} dx_i/dt = \rho[(x_i f_{x_i})(f_{xj} f_{xj}) - (f_{x_i} f_{x_i})(x_j f_{xj})] = 0.$$

Hence f is constant along the trajectory.

Let us consider the following:

$$\frac{dr}{dt} = \frac{x_i \dfrac{dx_i}{dt}}{r} = \frac{\rho[(x_i x_i)(f_{x_j} f_{x_j}) - (x_i f_{x_i})(x_j f_{x_j})]}{r}$$

$$= \frac{\rho(x_i f_{x_j} - x_j f_{x_i})(x_i f_{x_j} - x_j f_{x_i})}{2r} = 1,$$

as follows from (9.3). Hence t may be taken equal to r. This completes the proof.

10. *The locus $f = $ constant in the neighborhood of a critical point of intermediate type.*

THEOREM 6. *Let P be a critical point of intermediate type at which $f = c$. Let S be an $(n-1)$-sphere with center at P, so small that (9.1) is satisfied at all points of the locus $f = c$ within or on S. Then the real intersection of S with the locus $f = c$ consists of a finite number of regular analytic $(n-2)$-dimensional manifolds.*

Proof. Let B denote the set of intersection points in question. The points of B are given by the real simultaneous solutions of the equations

$$\phi \equiv f - c = 0, \tag{10.1}$$

$$\psi \equiv (x_i x_i) - d^2 = 0, \tag{10.2}$$

where d is the radius of the sphere S. The Jacobians of these functions are

$$[D(\phi, \psi)/D(x_i, x_j)] = 2(x_j f_{x_i} - x_i f_{x_j}).$$

By hypothesis (9.1) is satisfied. Therefore not all these Jacobians are zero at any point of B. Hence * the points of B neighboring a fixed point of B are given by expressing two of the variables as real analytic functions of the other $n-2$. Consequently any connected part of B is a regular analytic $(n-2)$-spread. Since ϕ and ψ are continuous, it is a closed set of points. It can be shown to be a manifold.† The number of these manifolds is very easily shown to be finite.‡ Thus Theorem 6 is proved.

THEOREM 7. *Let P be a critical point of intermediate type, at which*

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* Osgood I, page 70, and II, page 14.

† The hypotheses at hand are sufficient to ensure that the locus is a complex in the sense of analysis situs. Similar results will be needed in section 11. The necessary proofs have been given by B. L. van der Waerden. For reference and proofs see Lefschetz's *Colloquium Lectures on Topology*, Chapter 8.

‡ Cf. Morse, *loc. cit.*, page 348.

$f = c$. *Let S be an $(n-1)$-sphere with center at P, so small that* (9.1) *is satisfied at all the points, say A, of the locus $f = c$ within or on S. Let B be the intersection of S with the locus $f = c$.*

Then A can be put in one-to-one continuous correspondence with the set of all points on straight line segments joining P to points of B.

Proof. Let a correspondence be set up in which a given line segment, say L, is made to correspond to the trajectory τ on A that cuts B in the same point as L. Corresponding points shall be equi-distant from P. By use of Lemma 11 this correspondence is easily proved to be continuous.

11. *Differences in Betti numbers of the loci $f \leqq c - e$ and $f \leqq c + e$.* We shall begin by introducing notations for certain complexes that we shall use.

Any value assumed by f at a critical point is called a critical value.

Let $e > 0$ and c be constants such that c is the only critical value in the interval $c - e \leqq f \leqq c + e$. Let a set of $(n-1)$-spheres S be taken, one about each critical popint of intermediate type on $f = c$ as center, small enough to satisfy the hypotheses of Theorem 7.

Let A be the complex of all points satisfying $f \leqq c$ within or on the spheres S.

Let B be the complex of all points satisfying $f \leqq c$ except those interior to the spheres S.

Let B' be the (generalized) complex obtained by removing from B the points on $f = c$ at which f has maxima and minima.

Let C be the complex of points on the spheres S satisfying $f \leqq c$.

Let D be the complex of all points satisfying $f \leqq c$.

Let D' be the complex obtained by removing from D all the critical points at which $f = c$.

Let E and F be the complexes satisfying $f \leqq c + e$ and $f \leqq c - e$ respectively.

Let H be the complex of all points satisfying $f = c$ within or on the spheres S.

LEMMA 12. *The complexes D and E have the same Betti numbers.*

Proof. Let e_1 be taken satisfying $0 < e_1 \leqq e$ and so small that the complex of points satisfying

$$c \leqq f \leqq c + e_1$$

can be deformed onto the locus $f = c$, through points satisfying

$$c \leqq f \leqq c + e.^*$$

Next we define a deformation in two steps. In the first step the points satisfying

$$c + e_1 < f \leqq c + e$$

move along their orthogonal trajectories so that f decreases at a constant rate; stopping when they reach the locus

$$f = c + e_1.$$

The second step is the deformation mentioned in defining e_1.

The deformation described in these two steps shows, by Theorem 2, that D and E have the same Betti numbers, as was to be proved.

LEMMA 13. *The complexes F and D' have the same Betti numbers.*

Proof. We make a deformation of D' along orthogonal trajectories, which moves only those of its points satisfying $c - e < f \leqq c$. This deformation may be defined in a manner similar to the first one used in the proof of Lemma 12, so as to carry these points onto the locus $f = c - e$. The deformation is possible, because there is a unique orthogonal trajectory through each point that is moved, leading to the locus $f = c - e$. We may now apply Theorem 2, and find that Lemma 13 is true.†

LEMMA 14. *The complexes B' and D' have the same Betti numbers.*

Proof. We shall make a deformation satisfying the hypotheses of Theorem 2, and carrying the points that are in D' but not in B', onto the spheres S. It will be sufficient to describe the deformation within a single one of the spheres S. Let P be the corresponding critical point.

Consider an $(n-1)$-sphere S' with center at P, starting with zero radius and then expanding at a constant rate till it coincides with S. The deformation is then defined as follows.

Any given point Q remains fixed till S' reaches that point. Thereafter Q travels on S' towards S along a fixed straight line through P, that is, so

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* The property of a complex here implied is proved in Lefschetz's Colloquium Lectures on "Topology."

† The idea of using D', obtained from D by the removal of the critical points, was suggested to me by Professor Lefschetz. It introduced a great simplification in my original treatment, and also has enabled me to state the final results in slightly simpler form.

long as it does not reach H. If Q reaches H it then travels along a trajectory τ (see Theorem 7) on H for such time as the points on the (moving) radial line through P and Q near Q but on the side away from P, are points at which $f > c$. (But Q still remains on S'.) As soon as the case arises that these points satisfy $f \leqq c$, then Q begins again to move along a fixed line through P.

If it happens that the radial line becomes tangent to H, then the trajectory τ through the point in question has the same direction as the radial line. Therefore the motion is continuous, and constitutes a deformation.

The deformation is seen to satisfy the hypotheses of Theorem 2. Therefore D' and B' have the same Betti numbers, and the lemma is proved.

LEMMA 15. *The differences in Betti numbers of the loci* $f \leqq c - e$ *and* $f \leqq c + e$ *are the same as the corresponding differences for* B' *and* D.

Proof. The result follows from Lemmas 12, 13 and 14, and the definitions of E and F.

LEMMA 16. *The Betti numbers of* A *are* $R_0 = N$, $R_1 = R_2 = \cdots = R_n = 0$; *where* N *is the number of critical points of intermediate type at which* $f = c$.

Proof. This is proved by making a deformation carrying A into the corresponding set of N critical points, and then applying Theorem 2. The deformation is defined in a manner similar to that used in the proof of Lemma 14, using contracting spheres instead of expanding spheres. Because of the similarity to the proof of Lemma 14, it is unnecessary to give any further details.

Let P be any critical point of intermediate type, and C_p the complex of points on the sphere S about P as center, at which $f \leqq c$. *The i-th type number,* $M_i(P)$, *of the given critical point, is defined by*

$$M_i(P) = R_{i-1}(C_p), \qquad (i = 2, 3, \cdots, n-1);$$
$$M_1(P) = R_0(C_p) - 1.$$

Thus each critical point of intermediate type has $(n-1)$ type numbers.

THEOREM 8. *The type numbers are well defined.*

Proof. We must show that the Betti numbers of C_p do not change when we take spheres S of different magnitudes. Let S' be a sphere whose radius differs little from that of S, and let C_p' be the corresponding part in question. We suppose that both S' and S satisfy the hypotheses of Theorem 7.

Let C_p'' be the region obtained by projecting C_p' radially on the sphere S. We now make a stereographic projection of S onto a tangent $(n-1)$-plane L, taking the pole at a point of S not in C_p or C_p''. We shall use the same notation for the projections of C_p and C_p'' as for the original regions.

Now C_p is an $(n-1)$-region in L, bounded by regular analytic $(n-2)$-spreads. (See Theorem 6.) We can deform the part of C_p near the boundary a short distance toward the interior along the normals to the boundary, till the deform is interior to C_p''. From Theorem 1 we conclude that the Betti numbers of C_p'' are at least as great as those of C_p. By symmetry it follows that the Betti numbers of C_p are at least as great as those of C_p''. Therefore the two have the same Betti numbers, and consequently the same is true of C_p and C_p' as originally defined.

We have proved that if S is one sphere which may be used, then spheres of radii sufficiently near that of S define complexes C_p having the same Betti numbers. It follows that all suitable spheres S give complexes C_p with the same Betti numbers. For if any two are given, we can show easily by use of the Heine-Borel theorem and the above result that we can pass from one to the other with no change in Betti numbers. Therefore Theorem 8 is proved.

THEOREM 9. *Let c be a critical value and $e > 0$ a suitably chosen small constant. Let ΔM_0 and ΔM_n be the numbers of points of minimum and maximum, respectively, at which $f = c$. Let ΔM_i, $i = 1, 2, \cdots, n-1$, be the sum of the i-th type numbers of all the critical points of intermediate type at which $f = c$. Let E and F be the complexes satisfying $f \leqq c + e$ and $f \leqq c - e$ respectively, and $\Delta R_i = R_i(E) - R_i(F)$. Then the following relations are valid:*

$$\Delta M_0 \geqq \Delta R_0,$$
$$\Delta(M_0 - M_1) \leqq \Delta(R_0 - R_1),$$
$$\Delta(M_0 - M_1 + M_2) \geqq \Delta(R_0 - R_1 + R_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta[M_0 - M_1 + \cdots + (-1)^n M_n] = \Delta[R_0 - R_1 + \cdots + (-1)^n R_n].$$

Proof. We shall evaluate the right hand members of the above relations. According to Lemma 15 the changes in Betti numbers are those obtained when the points of minimum and maximum are added to B', and then the complex A is added (except for C, which is already part of B').

Now the minima are not connected to B', and the maxima have all their neighboring points belonging to B'. It follows from Lemmas 7 and 8 that the addition of the minima and maxima contributes M_0 to ΔR_0 and $(-M_n)$ to ΔR_{n-1}. The addition of the maxima and minima converts B' into B.

Next we add the complex A (except for the part of its boundary already in B). The resulting complex is D. The points common to A and B form the complex C. Here we can apply the second corollary to Theorem 3, at the end of section 4. The A, B, C and D of Theorem 3 will be the complexes of the same names defined at the beginning of this section. We begin by giving certain formulas and substitutions to be used in the application of the corollary.

In view of the result just obtained, we have the following formulas for the ΔR_i:

$$\Delta R_0 = \Delta M_0 + D_0 - B_0,$$
$$\Delta R_i = D_i - B_i, \qquad i = 1, 2, \cdots, n-2, n.$$
$$\Delta R_{n-1} = D_{n-1} - B_{n-1} - \Delta M_n.$$

We find from Lemma 16 that $A_0 = N$, $A_1 = A_2 = \cdots = A_n = 0$. Here N denotes the number of critical points of intermediate type at which $f = c$.

From the definition of type number we conclude that

$$C_0 = \Delta M_1 + N,$$
$$C_i = \Delta M_{i+1}, \qquad i = 1, 2, \cdots, n-2.$$

Since C is a sum of parts of $(n-1)$-spheres, but contains no complete $(n-1)$-spheres, we know that $C_{n-1} = C_n = 0$.

The n-th Betti numbers of all the complexes are zero, since none of them fills out n-space.

We are now ready to use the right hand set of relations in the corollary, but we replace the first relation by the following:

$$A_0 + B_0 \geqq D_0 + N.$$

To prove this relation we observe that A contains N parts, each of which is connected to B. That makes the inequality obvious, since the zeroth Betti number of a complex equals the number of paprts of the complex.

After making the replacement just mentioned, we solve the right hand relations of the corollary for values of the right hand members of the relations stated in Theorem 9. By using the facts stated above, we find that the result is the set of relations forming the conclusion of the theorem. This completes the proof.

12. *Relations between the critical points.* Morse proved* that if there is no critical value in the interval $a \leqq f \leqq b$, $(a < b)$, then the com-

* Morse, *loc. cit.*, pp. 358-360, and 396.

plexes defined by the relations $f \leqq a$ and $f \leqq b$ have the same Betti numbers. Since the Betti numbers of an empty complex are all zero, it follows that relations involving the Betti numbers of the region R over which f is defined may be obtained by adding the relations of Theorem 9 for all critical values. Thus we get the following theorem.

THEOREM 10. *Let M_0 and M_n be the numbers of critical points of minimum and maximum, respectively, of f in R. Let M_i be the sum of the i-th type numbers of all the critical points of intermediate type in R. Let R_i denote the i-th Betti number of R. Then the following relations hold.*

$$M_0 \geqq R_0,$$
$$M_0 - M_1 \leqq R_0 - R_1,$$
$$M_0 - M_1 + M_2 \geqq R_0 - R_1 + R_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$M_0 - M_1 + \cdots + (-1)^n M_n = R_0 - R_1 + \cdots + (-1)^n R_n.$$

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Numbers of Representations in Certain Quinary Quadratic Forms.

By E. T. Bell.

1.1. *Introduction.* A general problem, indicated in §§ 1.4, 1.5, concerning enumerations of representations in quadratic forms is illustrated by complete enumerations for four forms in 5 variables. Several partial enumerations for similar forms are determined incidentally while discussing the four forms; these results will be omitted here. These apparently are the first instances of enumerations for quinary quadratic forms other than a sum of 5 squares. The paper also illustrates several simple, practicable methods which I have applied successfully to forms in 2, 3, 4, 5, $\cdots$, 15 variables. Selections from the numerous theorems thus obtained will be published later.

1.2. *Notation.* Without further reference, α is an integer $\geqq 0$; n is an integer > 0; m is an odd integer > 0; μ, μ_1, $\mu_2 \cdots$ are odd integers $\geqq 0$; $x, y, z, w, u, v, v_1, v_2, \cdots$ are integers $\gtreqless 0$. Restrictions consistent with any of the foregoing will always be indicated, as, for example, $m \equiv 3 \bmod 4$; such restrictions will apply only to the equation before which they are written. The number of distinct one-rowed matrices $(x, y, z, \cdots)$ such that $n = f(x, y, z, \cdots)$ is the number of representations of n in $f(x, y, z, \cdots)$, and is denoted by $N[n = f(x, y, z, \cdots)]$ or, when there can be no confusion, by $N[n = f]$. We define $\phi_\alpha(n)$ by

$$\phi_\alpha(n) \equiv N[n = x^2 + y^2 + z^2 + w^2 + 2^\alpha u^2].$$

Hence $\phi_0(n)$, which will be written $\phi(n)$, is the number of representations of n as a sum of 5 squares. The functions $\phi_\alpha(n)$, $\alpha > 0$, are fundamental for the work mentioned in § 1.1, end.

1.3. *Simple and Compound Enumerations.* The general and, as yet, unsolved problem in § 1 can be precisely formulated. It will be simpler, however, for the present, merely to describe the nature of the problem with precision sufficient to cover the forms actually treated in this paper, which will serve as an introduction to a more general statement.

Let f be a quadratic form such that $N[n = f]$ is finite. All $N[n = f]$ fall into two mutually exclusive classes: all $N[n = f]$ that are polynomials in the real divisors of cn alone, where c is an integer not depending upon n;

all other $N[n=f]$. Members of the first class will be called *simple;* of the second, *compound.*

The classification of compound enumerations can be carried out in several sufficient (but usually unsatisfactory) ways. For example, if the generating function of $N[n=f]$ is a modular invariant, this invariant may be expressed in terms of the members of a fundamental set, and the enumerative functions generated by the members of any such set may be taken as the elementary functions of which $N[n=f]$ is a function. This particular classification throws no light on the arithmetical properties of the $N[n=f]$ concerned.

1.4. *The Function* $\phi(n)$. It is known that $N[n=f]$ is simple when f is a sum of 2, 4, 6 or 8 squares, and compound when n is a sum of $2r (r>4)$ squares, or when n is a sum of 3, 5, 7, $\cdots$ squares. Further, $\phi(n)$ is not rationally expressible in terms of the number of representations of n as a sum of 3 squares, or a sum of $2r (r>4)$ squares. Hence if a particular $N[n=f]$ is determined in the form

$$a_1\phi(b_1n)+a_2\phi(b_2n)+\cdots+a_s\phi(a_sn),$$

where $s, b_1, \cdots, b_s$ are integers >0 independent of n, and $a_1, \cdots, a_s$ are rational numbers independent of n, it is futile to seek any essentially simpler expression for $N[n=f]$.

We therefore regard $\phi(n)$ as a new elementary enumerative function, and seek those forms f such that

$$N[n=f]=c\chi(an)+a_1\phi(b_1n)+\cdots+a_s\phi(b_sn)$$

where χ is simple, and $a>0$, $c \gtreqless 0$ are independent of n.

Similar statements may be made with respect to $\phi_\alpha(n)$, $\alpha>0$, or any $N[n=f]$ which can be shown to be compound.

1.5. *Degeneration.* Let $N[n=f]$ be compound. For n suitably restricted, the compound $N[n=f]$ may degenerate to a simple function. Thus, if f is a sum of 10 squares $N[n=f]$ is compound, but $N[4\alpha+3=f]$ is simple. Again, if f is a sum of 3 or of 5 squares, it was shown by Stieltjes and Hurwitz that $N[n^2=f]$ is simple, and a similar result has recently been proved for 7 squares by Dr. Gordon Pall, who has also generalized the theorems.*

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* In passing, it may be pointed out that it has not yet been proved for these $N[n=f]$ that no integers a, b exist such that $N[an+b=f]$ is simple. That is, it is not known whether there exist for 3, 5, 7, $\cdots$ squares analogues of the theorems for 10 and 12 squares. This indicates the undeveloped state of the theory.

The like is not true for 9 squares; the phenomenon reappears in one case of 11 squares, beyond which nothing concerning this point is known.

The foregoing remarks suggest the problem of determining the polynomial $P(n)$ of lowest degree with integral coefficients such that, when f is given and $N[n = f]$ is compound, $N[P(n) = f]$ is simple. For the forms discussed in this paper $P(n)$ is n^2.

It is interesting to observe the evident analogies between § 1. 4, § 1. 5 and classical problems of the integral calculus.

2. 1. *Complete Enumerations for Four Special Quinaries.* Before indicating the proofs, we summarize the results for the forms $f_4, \cdots, f_1$ where

$$\begin{aligned} f_4 &\equiv x^2 + 4y^2 + 4z^2 + 4w^2 + 4u^2, \\ f_3 &\equiv x^2 + y^2 + 4z^2 + 4w^2 + 4u^2, \\ f_2 &\equiv x^2 + y^2 + z^2 + 4w^2 + 4u^2, \\ f_1 &\equiv x^2 + y^2 + z^2 + w^2 + 4u^2. \end{aligned}$$

For f_4, we shall prove the following:

$$\begin{aligned} m \equiv 3 \bmod 4: &\quad N[m = f_4] = 0; \\ m \equiv 1 \bmod 8: &\quad 5N[m = f_4] = \phi(m); \\ m \equiv 5 \bmod 8: &\quad 7N[m = f_4] = \phi(m); \\ &\quad N[2m = f_4] = 0; \\ &\quad N[4n = f_4] = \phi(n). \end{aligned}$$

For f_3,

$$\begin{aligned} m \equiv 3 \bmod 4: &\quad N[m = f_3] = 0; \\ m \equiv 1 \bmod 8: &\quad 5N[m = f_3] = 2\phi(m); \\ m \equiv 5 \bmod 8: &\quad 7N[m = f_3] = 2\phi(m); \\ &\quad 10N[2m = f_3] = \phi(2m); \\ &\quad N[4n = f_3] = \phi(n). \end{aligned}$$

For f_2,

$$\begin{aligned} m \equiv 3 \bmod 4: &\quad 10N[m = f_2] = \phi(m); \\ m \equiv 1 \bmod 8: &\quad 5N[m = f_2] = 3\phi(m); \\ m \equiv 5 \bmod 8: &\quad 7N[m = f_2] = 3\phi(m); \\ &\quad 10N[2m = f_2] = 3\phi(2m); \\ &\quad N[4n = f_2] = \phi(n). \end{aligned}$$

For f_1,

$m \equiv 3 \bmod 4$: $\quad 5N[m = f_1] = 2\phi(m)$;

$m \equiv 3 \bmod 4$: $\quad 35N[2^{2a+2}m = f_1] = 3(2^{3a+4} + 5)\phi(m)$;

$m \equiv 1 \bmod 8$: $\quad 5N[m = f_1] = 4\phi(m)$;

$m \equiv 1 \bmod 8$: $\quad 35N[2^{2a+2}m = f_1] = (3 \cdot 2^{3a+5} - 5)\phi(m)$;

$m \equiv 5 \bmod 8$: $\quad 7N[m = f_1] = 4\phi(m)$;

$m \equiv 5 \bmod 8$: $\quad 49N[2^{2a+2}m = f_1] = 3(2^{3a+5} + 3)\phi(m)$;

$\quad\quad\quad\quad\quad\quad\quad\quad 35N[2^{2a+1}m = f_1] = 3(2^{3a+1} + 5)\phi(2m)$.

2.2. *Degeneracies.* For the following statement of Hurwitz' theorem on 5 squares, we refer to a former paper, where full references are given.* Denote by $\zeta_r(n)$ the sum of the r^{th} powers of all the divisors of n, and define $H(m)$ by

$$H(m) \equiv \Pi[\zeta_3(p^a) - p\zeta_3(p^{a-1})], \quad H(1) = 1,$$

where $m = \Pi p^a = p^a q^b \cdots r^c$ is the resolution of m into powers of distinct primes $p, q, \cdots, r$. Then Hurwitz' theorem may be written

$$\phi(2^{2a}m^2) = 10\zeta_3(2^a)H(m).$$

All the degeneracies for $f_4, \cdots, f_1$ obtainable by application of the foregoing are read off from § 2.1. Observe that in § 2.1 only the cases $m \equiv 1 \bmod 8$ and $N[4n = f]$ need be considered, since an odd square is $\equiv 1 \bmod 8$, and an even square is $\equiv 0$ or $4 \bmod 8$. Hence

$$N[m^2 = f_4] = 2H(m), \quad 7N[2^{2a+2}m^2 = f_4] = 10(2^{3a+3} - 1)H(m);$$
$$N[m^2 = f_3] = 4H(m), \quad 7N[2^{2a+2}m^2 = f_3] = 10(2^{3a+3} - 1)H(m);$$
$$N[m^2 = f_2] = 6H(m), \quad 7N[2^{2a+2}m^2 = f_2] = 10(2^{3a+3} - 1)H(m);$$
$$N[m^2 = f_1] = 8H(m), \quad 7N[2^{2a+2}m^2 = f_1] = 2(3.2^{3a+5} - 5)H(m).$$

Since every square is of one or other of the forms m^2, $2^{2a+2}m^2$, degeneration of $N[n = f]$ when n is a square is complete for each of the forms $f_4, \cdots, f_1$.

2.3. *Reduction Formulas.* Most of the statements in § 2.1 can be disposed of by observing the necessary and sufficient congruential forms modulo 4 of the numbers representable in a particular f. Passing these for the moment, we consider certain reduction formulas. Such seem to play an essential part in complete determinations of all but the simplest $N[n = f]$.

The forms $f_4, \cdots, f_1$ are in 5 variables with coefficients powers of 2. Hence it is suggested that we examine those theta identities of degree 5

* *Transactions of the American Mathematical Society*, Vol. 26 (1924), p. 448.

arising from transformations of order a power of 2. Similarly for forms in any number of variables at least one of whose coefficients is divisible by a prime p (other than 2). The forms $f_4, \cdots, f_1$ are too simple, however, to exhibit the power of this naturally suggested method, and we shall use a more direct means. The special simplification for $f_4, \cdots, f_1$ enters through the identity $\vartheta_0^4 + \vartheta_2^4 = \vartheta_3^4$, which is only one degree lower than those required. The notation is as usual

$$\vartheta_a \equiv \vartheta_a(q), \quad \vartheta_3(q) = \vartheta_0(-q) = \Sigma q^{\nu^2}, \quad \vartheta_2(q^4) = \Sigma q^{\mu^2}.$$

If the stated identity be multiplied in turn by ϑ_0, ϑ_2 ϑ_3 throughout, the three resulting identities when interpreted arithmetically give practically all of the required information for the forms considered in this paper. We need attend only to one aspect of the identity

$$\vartheta_0^4\vartheta_3 + \vartheta_2^4\vartheta_3 = \vartheta_3^5$$

as the rest of the analysis is of the same nature. Similarly, for the identities deduced from these by the transformations of orders 2, 4, 8.

Comparison of coefficients in the given identity yields

$$[n = \nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2 + \nu_5^2 : \Sigma(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}] + N[4n = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\nu_5^2] = \phi(n),$$

where $[n = f : \Sigma$ etc.] denotes the indicated sum taken over all sets of solutions of $n = f$. Since precisely 0, 1, 2, 3 or 4 of ν_1, ν_2, ν_3, ν_4 may be odd, the last is equivalent to

$$\begin{aligned} & N[n = \nu_1^2 + 4\nu_2^2 + 4\nu_3^2 + 4\nu_4^2 + 4\nu_5^2] \\ + & 6N[n = \mu_1^2 + \mu_2^2 + 4\nu_3^2 + 4\nu_4^2 + 4\nu_5^2] \\ + & N[n = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\nu_5^2] \\ - & 4N[n = \mu_1^2 + 4\nu_2^2 + 4\nu_3^2 + 4\nu_4^2 + \nu_5^2] \\ - & 4N[n = \mu_1^2 + \mu_2^2 + \mu_3^2 + 4\nu_4^2 + \nu_5^2] \\ + & N[4n = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\nu_5^2] = \phi(n), \end{aligned}$$

in some lines of which obviously permissible changes of suffixes have been made. Change n into $4n$ and attend to the necessary congruential possibilities. Then

$$N[16n = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\nu_5^2] - 3n[4n = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\nu_5^2] = \phi(4n) - \phi(n),$$

a difference equation for

$$N[4n = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\nu_5^2].$$

Now in $4n = f_1$, necessarily

$$x^2 + y^2 + z^2 + w^2 \equiv 0 \bmod 4,$$

and therefore only precisely 0 or 4 of x, y, z, w may be odd. Thus

$$N[4n = f_1] = \phi(n) + N[4n = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\nu_5^2].$$

Hence if we write

$$\psi(\alpha) \equiv N[2^\alpha m = f_1], \quad \lambda(\alpha) \equiv \phi(2^\alpha m),$$

the difference equation is equivalent to

$$\psi(\alpha+4) - 3\psi(\alpha+2) = 2\lambda(\alpha+2) - 4\lambda(\alpha).$$

Suppressing the simple algebra, we write down the solution. Let $\{x\} = 0$ or the greatest integer $\leqq x$, according as $x \leqq 0$ or $x > 0$. Then the solution is

$$\psi(\alpha+4) - 3^{\{\alpha/2\}+1}\psi(\alpha - 2\{\alpha/2\} + 2)$$
$$= 2\lambda(\alpha+2) - 4\cdot 3^{\{\alpha/2\}}\lambda(\alpha - 2\{\alpha/2\}) + 2\sum_{j=0}^{\{\alpha/2\}-1} 3^j \lambda(\alpha - 2j),$$

the summation being absent when $\alpha < 2$.

In general, it is difficult or impossible to proceed beyond this stage. Hence it may be pointed out that the solution implied for the number of representations of $2^{\alpha+4}m$ in f_1 is fairly satisfactory as it depends only upon the like for $4m$ or $8m$, according as α is even or odd, and a sum of functions ϕ whose arguments depend only upon α (not upon m). However, in this case the solution can be completed by means of the following very special cases of Dr. Pall's theorems *:

$$\begin{aligned}
m \equiv 3 \bmod 4: &\qquad 7\lambda(2\alpha) = (2^{3\alpha+2} + 3)\phi(m);\\
m \equiv 1 \bmod 8: &\qquad 7\lambda(2\alpha) = (2^{3\alpha+3} - 1)\phi(m);\\
m \equiv 5 \bmod 8: &\qquad 49\lambda(2\alpha) = (5.2^{3\alpha+3} + 9)\phi(m);\\
&\qquad 7\lambda(2\alpha+1) = (2^{3\alpha+2} + 3)\phi(2m).
\end{aligned}$$

For the complete reduction, we shall need also the following, whose origin will be indicated presently:

$$\begin{aligned}
m \equiv 3 \bmod 4: &\qquad 5\psi(2) = 9\phi(m);\\
m \equiv 1 \bmod 8: &\qquad 5\psi(2) = 13\phi(m);\\
m \equiv 5 \bmod 8: &\qquad 7\psi(2) = 15\phi(m).
\end{aligned}$$

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* To be published in the *Journal of the London Mathematical Society*.

If the foregoing be applied to the solution above for $\psi(\alpha+4)$, after separation of the cases α even, α odd, all of the summary in 2.1 for $N[n=f_1]$, except the results for $2^\alpha m$ $(\alpha=0,\ 3)$, is found at once. Passing these for the moment, all is as stated for f_1.

2.4. *Remaining Cases.* In $2n=f_4$, x must be even, and $2m=f_4$ is impossible. In $m=f_4$, necessarily $m\equiv 1 \bmod 4$. Now in $m=x^2+y^2+z^2+w^2+u^2$, only precisely 5, 3 or 1 of x, y, z, w, u may be odd. Hence

$$\begin{aligned}\phi(m)=&N[m=\mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2+\mu_5^2]\\&+10N[m=\mu_1^2+\mu_2^2+\mu_3^2+4\nu_4^2+4\nu_5^2]\\&+\ 5N[m=\mu_1^2+4\nu_2^2+4\nu_3^2+4\nu_4^2+4\nu_5^2],\end{aligned}$$

the coefficients 1, 10, 5 being supplied since the odd variables in the indicated fixed positions in the representations as a sum of 5 squares may be chosen from the available 5 in these numbers of ways. If $m\equiv 3 \bmod 4$ only the second N is present, and hence $10N[m=f_2]=\phi(m)$ if $m\equiv 3 \bmod 4$. If $m\equiv 1 \bmod 8$, only the third N is present, so that $5N[m=f_4]=\phi(m)$ when $m\equiv 1 \bmod 8$. Finally, if $m\equiv 5 \bmod 8$, only the first and third N's are relevant. This case can be disposed of in several ways. To illustrate one which (amplified) applies to forms in 3, 4, 5, $\cdots$, 15 variables, we cite a result which is written down immediately from two formulas in a previous paper *:

$$m\equiv 5 \bmod 8: \qquad 7N[m=\mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2+\mu_5^2]=2\phi(m),$$

which will often be found useful. From this and the above expression for $\phi(m)$, we find the result stated in § 2.1 for $N[m=f_4]$ when $m\equiv 5 \bmod 8$, which completes the discussion for f_4.

In $m=f_3$, precisely one of x, y must be even. Hence $N[m=f_3]=2N[m=f_4]$. Since in $4n=f_3$, both x and y must be even, $N[4n=f_3]=\phi(n)$. Again, since $2m\equiv 2 \bmod 4$, precisely 2 of the variables in $2m=x^2+y^2+z^2+w^2+u^2$ must be odd. Hence $10N[2m=f_3]=\phi(2m)$ as stated.

In $2n=f_2$, either precisely 0 or precisely 2 of x, y, z may be odd, and these exhaust the possibilities. If precisely 2 of x, y, z are odd, $n=m$. Hence $N[2m=f_2]=3N[2m=f_3]$. Clearly, $N[4n=f_2]=\phi(n)$. If $m=f_2$, then necessarily either precisely 1 or precisely 3 of x, y, z are odd. If the former, $m\equiv 1 \bmod 4$; if the latter, $m\equiv 3 \bmod 4$, and in particular

$$m\equiv 1 \bmod 8: \qquad N[m=f_2]=3N[m=f_4].$$

If $m\equiv 3 \bmod 4$, only precisely 3 of the variables in the representation of m

* *American Journal of Mathematics*, Vol. 42 (1920), pp. 177-178.

as a sum of 5 squares are odd. Thus $10N[m = f_2] = \phi(m)$ when $m \equiv 3 \bmod 4$. Finally, by an argument from congruences, as already frequently used, we find $m \equiv 1 \bmod 4$: $N[m = f_4] + N[m = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + \mu_5^2] = \phi(m)$; whence, by a similar argument, we get

$$m \equiv 1 \bmod 4\colon\ 5N[m = f_4] = \phi(m) - N[m = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + \mu_5^2];$$
$$m \equiv 1 \bmod 4\colon\ 5N[m = f_3] = 2\phi(m) - 2N[m = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + \mu_5^2];$$
$$m \equiv 1 \bmod 4\colon\ 5N[m = f_2] = 3\phi(m) - 3N[m = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + \mu_5^2];$$
$$m \equiv 1 \bmod 4\colon\ 5N[m = f_1] = 4\phi(m) - 4N[m = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + \mu_5^2].$$

Using the value determined above for the N on the right, we find from the third of these the remaining case of f_2, and from the fourth, the stated values of $N[m = f_1]$ for $m \equiv 1 \bmod 4$.

The value of $N[m = f_1]$ when $m \equiv 3 \bmod 4$ can be obtained in several ways. For brevity, we cite

$$m \equiv 3 \bmod 4\colon\quad \phi(m) = 20\Sigma\zeta_1(m - 4\nu^2) \quad (\nu = 0, \pm 1, \pm 2, \cdots)$$

from the former paper. Hence, by the theorem on 4 squares, the result follows.

There remains only $N[8m = f_1]$, which was deferred in § 2. 3. We shall prove the following in a moment:

$$2N[2n = \mu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2 + \nu_5^2] = N[8n = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\mu_5^2];$$
$$2N[m = x^2 + y^2 + z^2 + w^2 + 2u^2] = N[2m = x^2 + y^2 + z^2 + w^2 + u^2;\ x \text{ odd}];$$
$$2N[2m = f_3] = N[m = x^2 + y^2 + z^2 + w^2 + 2u^2].$$

In the first, take $n = m$, and recall in the last that $10N[2m = f_3] = \phi(2m)$. Then

$$5N[\ m = x^2 + y^2 + z^2 + w^2 + 2u^2] = \phi(2m);$$
$$5N[2m = x^2 + y^2 + z^2 + w^2 + u^2;\ x \text{ odd}] = 2\phi(2m);$$
$$5N[8m = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + \mu_5^2] = 4\phi(2m).$$

Again, in $8m = f_1$, only precisely 0 or 4 of x, y, z, w may be odd. Hence

$$N[8m = f_1] = \phi(2m) + N[8m = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + 4\mu_5^2],$$

as is easily seen from the residues mod 16. Hence from the last of the preceding, we have the value of $N[8m = f_1]$ as stated in 2. 1.

The first of the assumed theorems follows at once from

$$N_5(8n, 4) = 80\Sigma\zeta_1(2n - \mu^2),$$

quoted from the previous paper, *loc. cit.*, where $N_r(n, s)$ denotes the number of representations of n as a sum of r squares precisely s of which are odd. As a similar argument has already been used, we omit further details.

To vary the method of proof slightly for the second and third assumed theorems, and to bring out a historical point of some interest, we note that these theorems (with many more) are implicit in a result stated by Gauss in 1808.* To translate Gauss' formula to the present context, consider all resolutions of n into a pair of positive divisors t, τ of which τ is odd, $n = t\tau$, and write

$$\psi(n) \equiv \Sigma(-1)^t t,$$

the Σ extending to all such t. Let $\epsilon(n)=0$ or 1 according as n is not or is a square. Then the equivalent of Gauss' result is easily found to be the identity

$$\Sigma\psi(n-\nu^2) = -n\epsilon(n) \quad (\nu = 0, \pm 1, \pm 2, \cdots).$$

Recall that $24\zeta_1'(n)$, where $\zeta_1'(n)$ is the sum of all the odd divisors of n, is the number of representations of $2n$ as a sum of 4 squares, and observe that

$$\psi(m) = -\zeta_1'(m), \quad \psi(2n) = 2(-1)^n\psi(n).$$

Then, from the above identity, it follows immediately that

$$N[2n = x^2+y^2+z^2+w^2+u^2;\ x \text{ odd}] = 16n\epsilon(2n) + 8\Sigma\psi(2n-4\nu^2),$$
$$N[m = f_1] = 8m\epsilon(m) + 8\Sigma\psi(m-\mu^2),$$

the last Σ referring to $\mu = \pm 1, \pm 3, \pm 5 \cdots$. In the first, take $n = m$, $n = 2m$, and reduce the resulting identities by the above. Then

$$N[2m = x^2+y^2+z^2+w^2+u^2;\ x \text{ odd}] = 2N[m = x^2+y^2+z^2+w^2+2u^2],$$
$$N[4m = x^2+y^2+z^2+w^2+u^2;\ x \text{ odd}] = 8N[m = f_1].$$

From the first, by the argument from congruence as already often used, we find

$$N[m = x^2+y^2+z^2+w^2+2u^2] = 2N[2m = x^2+y^2+4z^2+4w^2+4u^2].$$

A similar argument on the second gives, on starting from the representation of $4m$ as a sum of 5 squares,

$$10N[m = f_1] = \phi(4m) - \phi(m);$$

whence, applying the reduction formulas for ϕ in § 2. 3, we have $N[m = f_1]$ as stated in § 2. 1.

If the detailed formulas used in the proofs, with the others in their theta sets as outlined in § 2. 3, be written out, it will be found that over a hundred enumerations for quinaries with coefficients 2, 4, 8, 16, $\cdots$, 128 are given with but little further labor.

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* *Werke*, Vol. 3, p. 434, first formula.

On Groups of Motion in Related Spaces.*

By M. S. KNEBELMAN.†

1. The object of this brief note is to prove two theorems pertaining to the groups of motion of two Riemann spaces which are in geodesic correspondence or are conformal to each other.

Let V_n be an n-dimensional space whose fundamental tensor has the components $g_{ij}{}^{(x)}$ in the coördinate system x. If $\xi^i(x)$ are the contravariant components of an infinitesimal motion of the space into itself, this vector must satisfy Killing's equations ‡

$$\xi_{i,j} + \xi_{j,i} = 0, \tag{1.1}$$

where ξ_i ($\equiv g_{ih}\xi^h$) are the covariant components of the motion and a subscript preceded by a comma denotes covariant differentiation with respect to the g's. The number of linearly independent solutions of the differential equations (1.1) can not exceed $n(n+1)/2$, spaces of constant Riemannian curvature being the only ones admitting a group of motions of this number of parameters.

We now suppose that another space $\bar{V}_n$, with fundamental tensor whose components in the coördinate system x are $\bar{g}_{ij}(x)$, is in geodesic correspondence with V_n. Then, as is well known,

$$\overline{\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \delta_j{}^i\phi_{,k} + \delta_k{}^i\phi_{,j}, \tag{1.2}$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ are the Christoffel symbols of the second kind formed of the g's, while $\overline{\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}}$ are the corresponding symbols formed of the $\bar{g}$'s; ϕ is a scalar which by contraction of (1.2) is found to be

$$\phi = [1/2(n+1)] \log (\bar{g}/g), \qquad g = |\, g_{ij} \,| \tag{1.3}$$

and $\delta_j{}^i$ is the Kronecker delta ($=1$ if $i=j$; $=0$ if $i \neq j$).

If $\bar{\xi}_i$ are the covariant components of a motion of V_n into itself, $\bar{\xi}_i$ must satisfy

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* Presented to the American Mathematical Society, September 6th, 1928.

† National Research Fellow.

‡ Cf. L. P. Eisenhart, *Riemannian Geometry*, Chap. VI. It is of course understood that the summation convention of a repeated index is used throughout.

$$\bar{\xi}_{i;j} + \bar{\xi}_{j;i} = 0 \tag{1.4}$$

where a subscript preceded by a semi-colon indicates covariant differentiation with respect to $\bar{g}_{ij}$. Now

$$\begin{aligned}\bar{\xi}_{i;j} &= (\partial\bar{\xi}_i/\partial x_j) - \bar{\xi}_h \overline{\left\{ \begin{matrix} h \\ ij \end{matrix} \right\}} \\ &= (\partial\bar{\xi}_i/\partial x_j) - \bar{\xi}_h \left\{ \begin{matrix} h \\ ij \end{matrix} \right\} - \bar{\xi}_j\phi_{,i} - \bar{\xi}_i\phi_{,j} \\ &= \bar{\xi}_{i,j} - \bar{\xi}_j\phi_{,i} - \bar{\xi}_i\phi_{,j}.\end{aligned}$$

Therefore equations (1.4) may be written as

$$(\bar{\xi}_{i,j} - 2\bar{\xi}_i\phi_{,j}) + (\bar{\xi}_{j,i} - 2\bar{\xi}_j\phi_{,i}) = 0$$

or

$$(\bar{\xi}_i e^{-2\phi})_{,j} + (\bar{\xi}_j e^{-2\phi})_{,i} = 0. \tag{1.5}$$

If we assume that the space V_n admits an r-parameter group of motions whose covariant components are $\xi_i^{(\alpha)}$, $\alpha = 1, \cdots, r$, an index in a parenthesis indicating the vector, then equation (1.5) admit just r linearly independent solutions which may be taken to be

$$\bar{\xi}_i^{(\alpha)} = a_\beta{}^\alpha \xi_i^{(\beta)} e^{2\phi} = a_\beta{}^\alpha \xi_i^{(\beta)} (\bar{g}/g)^{1/(n+1)}. \tag{1.6}$$

This proves

THEOREM I. *Every Riemannian space in geodesic correspondence with one which admits a given r-parameter continuous group of motions also admits an r-parameter continuous group of motions; the components of this group are linear combinations (with constant coefficients) of the components of the given group, multiplied by a scalar.*

As an immediate corollary of this theorem we have, since the only spaces admitting an $n(n+1)/2$-parameter group are spaces of constant curvature,

The only spaces that are in geodesic correspondence with a space of constant curvature are spaces of constant curvature,

a theorem due to Beltrami.*

2. We next suppose that V_n admits an r-parameter ($r < n$) group of motions and that $\bar{V}_n$ is conformal to V_n; i.g. $\bar{g}_{ij} = e^{2\sigma} g_{ij}$.

If $\xi_{(\alpha)}{}^i$, $\alpha = 1, \cdots, r$, are the contravariant components of the motions in V_n they must satisfy

* Cf. L. P. Eisenhart, *loc. cit.*, p. 134.

$$(2.1) \qquad g_{ih}(\partial\xi_{(a)}{}^h/\partial x_j) + g_{hj}(\partial\xi_a{}^h/\partial x^i) + \xi_{(a)}{}^h(\partial g_{ij}/\partial x^h) = 0,$$

which are Killing's equations in terms of contravariant components, and if $\xi_{(a)}{}^i$ are to be components of a motion in $\bar{V}_n$ we must have

$$(2.2) \qquad \xi_{(a)}{}^h(\partial\sigma/\partial x^h) = 0,$$

as follows from (2.1). We designate the operator $\xi_{(a)}{}^h(\partial/\partial x^h)$ by X_a; then since X_a are the generators of a group we have

$$(X_aX_\beta - X_\beta X_a)\sigma = c_{a\beta}{}^\gamma X_\gamma\sigma$$

$c_{a\beta}{}^\gamma$ being the structural constants of the group. Hence the system of differential equations (2.2) is complete and admits $n - r$ functionally independent integrals.

Hence we have

THEOREM II. *If a Riemannian space admits an r-parameter $(r < n)$ group of motions there exist $n - r$ functionally independent conformal spaces admitting the same group of motions.*

PRINCETON UNIVERSITY.

Determination of All Normal Division Algebras in Thirty-six Units of Type R_2.*

By A. Adrian Albert.†

1. *Introduction.* An element x of a normal division algebra A in n^2 units over a non-modular field F is said to have grade r if its minimum equation has degree r. If there exist k distinct elements $x, \theta_1(x), \cdots, \theta_{k-1}(x)$ which are polynominals in x and satisfy its minimum equation then x is said to have type R_k. Algebra A in n^2 units has type R_k if it contains an element x of grade n and type R_k.

In all of the papers ‡ on the determination of normal division algebras the algebras were first shown to be of type R_2 and then proved to be known algebras. For algebras A in n^2 units, n a prime, the assumption that A has type R_2 implies, by an almost trivial argument, that A has type R_n and is a cyclic algebra. But when n is composite the problem of showing A to be known is difficult even under the assumption that A has type R_2. We consider here the case $n = 6$ and show that all normal division algebras in thirty-six units of type R_2 are of type R_3 and are known in the broad sense recently defined.§ A necessary and sufficient condition that the known algebras be of type R_6 and hence of the kind constructed by L. E. Dickson is also given.

2. *Results presupposed.*¶ We shall assume the following known results for normal division algebras in thirty-six units over F.

Theorem 1. *Let x be an element of A. Any transform txt^{-1} of x by an element t of A is a root of the minimum equation of x and conversely, any root in A of the minimum equation of x is a transform of x.*

Theorem 2. *Let $B = CD$ where C and D are polynomials in w with*

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† National Research Fellow.

‡ Cf. *Transactions of the American Mathematical Society:* for algebras in nine units, Wedderburn, in Vol. 21 (1921), pp. 129-35; for algebras in sixteen units, Albert in Vol. 31 (1929), pp. 253-260.

§ A. A. Albert "On the Structure of Normal Division Algebras," *Annals of Mathematics*, Vol. 30 (1929), pp. 322-339.

¶ For Theorems 1 to 7 see "Structure of Normal Division Algebras" (*loc. cit.*). For Theorem 8 see the author's paper "Normal Division Algebras in 4 p^2 units, p an Odd Prime," *Annals of Mathematics*, Vol. 30 (1929), pp. 583-590.

coefficients in A. Let $\omega - y$ be a right divisor of B and let $D = Q(\omega - y) + R$ where $R \neq 0$ is free of ω. Then $\omega - RyR^{-1}$ is a right divisor of C.

Definition. x is said to have type S_k if x has type R_k and not type R_{k+1}.

THEOREM 3. *If x has type S_k and grade r then k is a divisor of r.*

THEOREM 4. *The grade of any element x of A is* 1, 2, 3 *or* 6.

THEOREM 5. *Let x be an element of A having grade six. Then the only elements of A commutative with x are polynomials in x with coefficients in F.*

THEOREM 6. *Let x be in A and have grade r and minimum equation $\phi(\omega) = 0$. Then there exist transforms $x_1 = x, x_2, x_3, \cdots, x_r$ of x such that*

$$\phi(\omega) \equiv (\omega - x_r)(\omega - x_{r-1}) \cdots (\omega - x_1). \tag{1}$$

Also the factors in $\phi(\omega)$ may be permuted cyclically.

THEOREM 7. *Let A contain either a cyclic sub-algebra of order nine over F or a generalized quaternion algebra over F. Then A has type R_6.*

THEOREM 8. *Let A contain x of grade six and type S_2. Then $F(x)$ contains an element y of grade six and with minimum equation*

$$\phi(\omega) \equiv \omega^6 + \alpha\omega^4 + \beta\omega^2 + \gamma = 0 \quad (\alpha, \beta, \gamma \text{ in } F). \tag{2}$$

3. *Algebras containing an element of grade* 3 *commutative with one of its transforms.* Let q_1 have grade 3 and minimum equation

$$\phi(\omega) \equiv \omega^3 + \alpha\omega^2 + \beta\omega + \gamma = 0 \quad (\alpha, \beta, \gamma \text{ in } F). \tag{3}$$

Let $q_2 = tq_1t^{-1} \neq q_1$ be a transform of q_1 and such that $q_2q_1 = q_1q_2$. Let first q_2 be a polynomial in q_1 with coefficients in F. Then (3) is a cyclic equation with

$$\theta(q_1) = tq_1t^{-1},\ \theta^2(q_1) = t^2q_1t^{-2},\ \theta^3(q_1) = t^3q_1t^{-3} = q_1$$

and the elements

$$q_1^i t^j (i, j = 0, 1, 2) \tag{4}$$

are linearly independent with respect to $F(t^3)$ and form a cyclic algebra T of order 9 over $F(t^3)$. The order of T with respect to F is $9a$ where a is the order of $F(t^3)$ and is a divisor of thirty-six, the order of A. If t^3 is in F then T is a cyclic algebra of order nine over F and A has type R_6 by theorem 7. Since t has at most grade six the order of T is at most eighteen, and if $F(t^3)$ has order greater than unity it is a quadratic field. Let t^3 be not in F and let r generate $F(t^3)$ such that $r^2 = \xi$ in F. Then $x = rq_1$ has grade six

since $x^2 = \xi q_1^3$ is in the cubic field $F(q_1)$ and is not in F, while x is not in $F(q_1)$ since r is not in $F(q_1)$, a cubic field thus having no quadratic subfield $F(r)$. The element x has type R_6 since if the minimum equation of ξq_1^2 is $\psi(\omega) = 0$, then the minimum equation of x is $\psi(\omega^2) = 0$ with roots rq_1, rq_2, $r(-\alpha - q_1 - q_2)$, $-rq_1$, $-rq_2$, $-r(-\alpha - q_1 - q_2)$ all of which are polynomials in x. In this case A has type R_6.

Next let q_2 be not a polynomial in q_1. Let $q_3 = -\alpha - q_1$. Consider the field $F(q_1, q_2)$. Its order is greater than three since q_2 is not in $F(q_1)$ a cubic sub-field of $F(q_1, q_2)$, so that $F(q_1, q_2)$ has order six and contains an element x of grade six generating it. Since x is not in $F(q_1)$, the elements

$$1,\ q_1,\ q_1^2,\ x,\ q_1x,\ q_1^2x \tag{5}$$

are linearly independent and form a basis of $F(x)$, so that every element of $F(x)$ is expressible in the form $a + bx$ where a and b are in $F(q_1)$. In particular $x^2 = c + dx = b - a^2 - 2ax$ for $\alpha = -2a$, $c = b - a^2$ and thus $(x + a)^2 = b$. Let $s = x + a$. We have $s^2 = b$ in $F(q_1)$ while $s = x + a$ is not in $F(q_1)$.

Let $b = \epsilon$ in F. Let $\mu_1 = sq_1$. Then $\mu_1^2 = \epsilon q_1^2$ an element of grade 3 in $F(q_1)$ while μ_1 is not in $F(q_1)$ and thus $F(\mu_1)$ has order greater than three and μ_1 has grade six. Also if the minimum equation of μ_1^2 is $\psi(\omega) = 0$ then the minimum equation of μ_1 is $\psi(\omega^2) = 0$. The field $F(\mu_1)$ is equal to $F(x)$ since s and q_1 are both in $F(x)$ while the orders of $F(x)$ and $F(\mu_1)$ are the same. Thus $\mu_2 = sq_2$ and $-\mu_2$ are both polynomials in μ_1. But $\mu_2^2 = \epsilon q_2^2$ is a transform of μ_1^2 and is a root of $\psi(\omega) = 0$ so that μ_2 and $-\mu_2$ are roots of the minimum equation of μ_1. Thus $\psi(\omega^2) = 0$ has the roots μ_1, μ_2, $-\mu_1$, $-\mu_2$. These elements are distinct since if $\mu_2 = \pm\mu_1$ then $q_2 = \pm q$, contrary to the hypothesis that q_2 is not a polynomial in q_1. Hence μ_1 has grade 6 and type R_4 and, by theorem 3 has type R_6 so that A has type R_6.

Consider next b not in F. Then b has grade three and since $s^2 = b$ and s is not in $F(b) = F(q_1)$, the grade of s is greater than three and is six. Also if the minimum equation of b is $\psi(\omega) = 0$ then the minimum equation of s is $\psi(\omega^2) = 0$ and $-s$ is a root of it. The elements

$$1,\ q_1,\ q_1^2,\ q_2,\ q_1q_2,\ q_1^2q_2 \tag{6}$$

are linearly independent and form a basis of $F(q_1, q_2) = F(s)$ since q_2 is not in the cubic field $F(q_1)$. Thus there exist elements g and h in $F(q_1)$ such that $s = g + kq_2$. Let $s_2 = g(q_2) + h(q_2)q_3$. We may write (3) in the form

$$\phi(\omega) \equiv (\omega - q_3)(\omega - q_2)(\omega - q_1). \tag{7}$$

Comparing coefficients in (3) and (7) we obtain

$$q_3 = -(q_1 + q_2 + \alpha), \quad q_3q_2 + q_3q_1 + q_2q_1 = \beta. \tag{8}$$

Substitute the value of q_3 in the second equation of (8). We obtain the relation

$$q_2^2 = -(q_1 + \alpha)q_2 - (\beta + q_1^2 + \alpha q_1). \tag{9}$$

But $s^2 = g^2 + 2ghq_2 + h^2q_2^2 = b$ in $F(q_1)$. Thus $-h^2(q_1 + \alpha) + 2gh = 0$. Since s is not in $F(q_1)$, $h \neq 0$ and $2g = h(q_1 + \alpha)$, so that $2s = h(q_1 + \alpha + 2q_2)$, and

$$\begin{aligned} 4s^2 &= h^2[4q_2^2 + 4(q_1 + \alpha)q_2 + q_1^2 + 2\alpha q_1 + \alpha^2] \\ &= h^2[-4(\beta + q_1^2 + \alpha q_1) + q_1^2 + 2\alpha q_1 + \alpha^2] = 4\sigma(q_1). \end{aligned}$$

By formally replacing q_1 by q_2 and q_2 by q_3 in the above we obtain $s_2^2 = \sigma(q_2)$. But $\sigma(q_2)$ is a transform of $\sigma(q_1) = b$ and is a root of $\psi(\omega) = 0$. Thus s_2 is a root of $\psi(\omega^2) = 0$. If $s_2 = \pm s$, then $s_2^2 = s_1^2 = \sigma(q_1) = \sigma(q_2)$. Since b is not in F and thus generates $F(q_2)$ this implies that q_2 is a polynomial in $\sigma(q_2) = \sigma(q_1)$ and hence in q_1, a contradiction of our hypothesis. Thus $\psi(\omega^2) = 0$ has the distinct roots s_2, $-s_2$, s, $-s$ and s has type R_4. Thus s has type R_6 and A has type R_6. This proves the following theorem.

THEOREM 9. *Let A be a normal division algebra in thirty-six units containing an element q of grade three which is commutative with one of its transforms not equal to itself. Then A has type R_6.*

4. *Algebras containing an element q_1 commutative with no one of its transforms.* Let q_1 have grade 3 and minimum equation

$$\phi(\omega) \equiv \omega^3 + \alpha\omega^2 + \beta\omega + \gamma = 0 \qquad (\alpha, \beta, \gamma \text{ in } F). \tag{10}$$

We may write

$$\phi(\omega) \equiv (\omega - q_3)(\omega - q_2)(\omega - q_1) = 0, \tag{11}$$

where q_3, q_2, q_1 are transforms of q, and $q_3q_1 \neq q_1q_3$, $q_2q_1 \neq q_1q_2$. Comparing coefficients in (10) and (11) we have

$$q_3q_2 + q_3q_1 + q_2q_1 = \beta. \tag{12}$$

The factors in (11) may be permuted cyclically and we may show that $x = q_3q_2 - q_2q_3 = q_1q_3 - q_3q_1 = q_2q_1 - q_1q_2 \neq 0$ by means of (12) and

$$q_2q_1 + q_2q_3 + q_1q_3 = q_1q_3 + q_1q_2 + q_3q_2 = p. \tag{13}$$

Applying theorem 2 with $y = q_2$, $B = CD \equiv \phi(\omega)$, $C = \omega - q_2$, $D = (\omega - q_2)(\omega - q_1)$, $R = q_2q_1 - q_1q_2 \neq 0$, we show that $xq_2x^{-1} = q_3$ and similarly by cylic permutation of the factors of (11), $xq_3x^{-1} = q_1$, $xq_1x^{-1} = q_2$, so that $x^3q_1x^{-3} = q_1$ and $x^3 = d$ is commutative with each of q_1, q_2, q_3. Let S be the algebra composed of all polynomials in q_1 and q_2 with coefficients in F. S contains x and thus q_3. The quantity d is commutative with each element of S. Let K be the field composed of all elements of S commutative with every element of S. Then d is in K and S is a normal division algebra over K.

LEMMA 1. *Let $\Gamma(\omega) = 0$ be the minimum equation of x with respect to K. Then, if $x_2 \equiv (\alpha + 3q_2)x(\alpha + 3q_2)^{-1}$, and we have the decomposition*

$$\Gamma(\omega) \equiv Q(\omega)(\omega - x_1), \tag{14}$$

$\omega - x_2$ is a right divisor of $Q(\omega)$.

Proof. $\omega - (q_1 - q_2)x(q_1 - q_2)^{-1}$ is a right divisor of $\Gamma(\omega) = 0$ since $q_1q_2 \neq q_2q_1$ and thus $q_1 - q_2 \neq 0$. It follows that if

$$\begin{aligned} R &= (q_1 - q_2)x(q_1 - q_2)^{-1} - x \\ &= [(q_1 - q_2)x - x(q_1 - q_2)](q_1 - q_2)^{-1} \\ &= (q_1 - q_2 - q_2 + q_3)x(q_1 - q_2)^{-1} \neq 0, \end{aligned}$$

then $\omega - R(q_1 - q_2)x(q_1 - q_2)^{-1}R^{-1}$ is a right divisor of $Q(\omega)$. But $q_1 + q_3 = -\alpha - q_2$, so that $R = -(\alpha + 3q_2)x(q_1 - q_2)^{-1} \neq 0$ since q_2 is not in F, and $R(q_1 - q_2)x(q_1 - q_2)^{-1}R^{-1} = (\alpha + 3q_2)x(\alpha + 3q_2)^{-1}$, proving the lemma.

LEMMA 2. *x has not grade 2 with respect to K.*

Proof. Let x be a root of

$$\psi(\omega) \equiv \omega^2 + \xi\omega + \eta \qquad (\xi, \eta \text{ in } K). \tag{15}$$

Then

$$\psi(\omega) = (\omega + \xi + x)(\omega - x).$$

Applying lemma 1 we have

$$(\alpha + 3q_2)x(\alpha + 3q_2)^{-1} = -\xi - x,$$

and

$$(\alpha + 3q_2)^2x(\alpha + 3q_2)^{-2} = x.$$

But $x(\alpha + 3q_2)^2 = (\alpha + 3q_3)^2x$ so that $(\alpha + 3q_2)^2 = (\alpha + 3q_3)$.[2] Since $(\alpha + 3q_2)^2$ is not in F and generates $F(q_2)$ of order 3, q_2 is a polynomial in

$(\alpha + 3q_2)^2$ and in $(\alpha + 3q_3)^2$. Thus $q_2q_3 = q_3q_2$ a contradiction since $x = q_3q_2 - q_2q_3 \neq 0$, and the lemma is proved.

Let x_2 be defined as in lemma 1. Then

$$\psi(\omega) \equiv \omega^3 - d \equiv (\omega - x_3)(\omega - x_2)(\omega - x_1), \tag{16}$$

where $x_1 = x$, x_3 is some transform of x. We shall first consider the case $x_1x_2 \neq x_2x_1$ and consequently $y = x_2x_1 - x_1x_2 \neq 0$.

By comparing coefficients in (16) we may show that $y = x_3x_2 - x_2x_3 = x_1x_3 - x_3x_1$. We also have $yx_1y^{-1} = x_2$, $yx_2y^{-1} = x_3$, $yx_3y^{-1} = y^3x_1y^{-3} = x_1$ and y^3 is commutative with x_1, x_2, x_3. Let $z_1 = x_1y$, $z_2 = x_1z_1x_1^{-1} = x_1^2yx_1^{-1}$ and $y^3 = \nu$. Let T be the algebra composed of all polynomials in x_1 and z_1. Then T contains $y = x_1^{-1}z$, $x_2 = yx_1y^{-1}$, $x_3 = yx_2y^{-1}$. The element $d = x_1^3$ is commutative with x_1 and x_2 and consequently with $y = x_2x_1 - x_1x_2$ and $z_1 = x_1y$. Hence d and ν are both commutative with every element of T. Let H be the field of all elements of T commutative with every element of T. The field H contains d and ν and T may be considered as a normal division algebra over H. By (16), $x_3^3 = d = x_3x_2x_1$ and $x_3^2 - x_2x_1 = 0$. But

$$z_1z_2 - z_2z_1 = x_1(yx_1^2y)x_1^{-1} - x_1^2y^2 = x_1(yx_1^2y - x_1y^2x_1)x_1^{-1}$$

and

$$\begin{aligned} x_3^2 - x_2x_1 &= y^2x_1^2y^{-2} - yx_1y^{-1}x_1 \\ &= yx_1^2(1/\nu)y - yx_1y^2(1/\nu)x_1 = y(yx_1^2y - x_1y^2x_1)(1/\nu) = 0 \end{aligned}$$

so that $yx_1^2y - x_1y^2x_1 = 0$ and $z_1z_2 = z_2z_1$ since $y \neq 0$ and $1/\nu \neq 0$ is commutative with x_1 and y. Let $z_3 = x_1^2z_1x_1^{-2} = x_1^3yx_1^{-2} = yx_1$. Then $z_1z_3 - z_3z_1 = x_1y^2x_1 - yx_1^2y = 0$ and $z_2z_3 - z_3z_2 = x_1(z_1z_2 - z_2z_1)x_1^{-1} - 0$. But, since $z_1 = x_1y$, $z_1^2 = x_1x_2y^2$,

$$z_1^3 = x_1x_2y^2x_1y = x_1x_2x_3\nu = \nu(x_2x_1 - y)x_3 = \nu(x_2x_1x_3) - \nu x_1y = \nu(d - z_1),$$

z is a root of

$$\omega^3 + \nu\omega - \nu d = 0. \tag{17}$$

LEMMA 3. *The element z_1 has grade 3 with respect to H.*

Proof. The grade of z_1 with respect to H is < 3 since z_1 satisfies (17). Now z_1 has not grade 1 with respect to H since z_1 is not commutative with x_1, an element of T. Let z_1 satisfy

$$P(\omega) \equiv \omega^2 + \lambda\omega + \mu = 0 \qquad (\lambda, \mu \text{ in } H). \tag{18}$$

Then $P(\omega) \equiv (\omega + \lambda + z_1)(\omega - z_1)$. But $\omega - z_2$ is a right divisor of and is commutative with $(\omega - z_1)$ so that $(\lambda + z_1) = -z_2 = -x_1 z_1 x_1^{-1}$,

$$x_1 z_2 x_1^{-1} = z_3 = -x_1(\lambda + z_1)x_1^{-1} = -\lambda x_1 x_1^{-1} + \lambda + z = z_1,$$

$$z_3 = y x_1 = z_1 = x_1 y,$$

a contradiction. This proves the lemma.

Hence the minimum equation of z_1 with respect to H is

$$\psi(\omega) \equiv \omega^3 + v\omega - vd = 0. \tag{19}$$

Consider the field $F(z_1, z_2, z_3)$. If its order is not six then it is three by lemma 3, and z_2 and z_3 are transforms of z_1 which are commutative with z_1, an element of grade 3. Hence, by the work of section 3, A has type R_6. If $F(z_1, z_2, z_3)$ has grade 6 then there exists an element t of grade six in $F(z_1, z_2, z_3)$ generating the field, and t has type R_3 since $x_1 t x_1^{-1}$, $x_1^2 t x_1^{-2}$, t are distinct and are in $F(t) = F(z_1, z_2, z_3)$.

It remains to consider the case $x_2 x_1 = x_1 x_2$. If x_1 has grade three then, since x_2 is a transform of x_1 commutative with x_1, A has type R_6 by the work of the preceding section. If x_1 has not grade three then, since its grade with respect to H is three, its grade with respect to F is greater than three and is six. Thus x_2 is a polynomial in x and $x_3 x_1 = x_1 x_3$ so that x_3 is a polynomial in x_1 and x_3, x_2, x_1 are all roots of the minimum equation of x_1 and are distinct by (16) and lemma 1. Hence x_1 has grade 6 and type R_3 and we have obtained the theorem

THEOREM 10. *Let A be a normal division algebra in thirty-six units containing an element of grade three. Then A has type R_3.*

5. *Algebras of type R_2.* Let A have type R_2. Then A contains an element x of grade six and type R_2 so that x has type R_3 or type S_2. If x has type S_2 then, by theorem 8, $F(x)$ and consequently A contains an element y of grade six and minimum equation $\phi(\omega^2) = 0$, so that y^2 has grade 3 and, by theorem 10, A has type in R_3. This proves

THEOREM 11. *Every normal division algebra in thirty-six units of type R_2 has type R_3.*

In the author's paper on "Algebras in $4p^2$ Units, p an odd Prime" (*loc. cit.*), it was shown that all normal division algebras of type R_3 were known, that is contained an element i whose minimum equation

$$(20) \qquad \phi(\omega) \equiv \omega^6 + \alpha\omega^4 + \beta\omega^3 + \gamma\omega^2 + \delta\omega + \epsilon = 0 \qquad (\alpha, \beta, \gamma, \delta, \epsilon \text{ in } F)$$

had type R_6 or S_3. If (15) has type S_3, that is there exists a polynomial in i, $\theta(i)$, such that $i = \theta^3(i)$, $\theta(i)$, $\theta^2(i)$ are roots of (15) and no other polynomial in i is a root of 15, then A was shown to have the following structure:

A contains $\mathfrak{Z}$ which is a cyclic algebra of order 9 over the quadratic field $F(q)$ of all polynomials in i which are symmetric in the functions i, $\theta(i)$, $\theta^2(i)$. The algebra $\mathfrak{Z}$ has a basis

$$(21) \qquad 1, i, i^2, y, iy, i^2y, y^2, iy^2, i^2y^2$$

and the multiplication table given by (15) for $\omega = i$, and

$$(22) \qquad y^\lambda f(i) = f[\theta^\lambda(i)]y^\lambda \qquad (\lambda = 0, 1, 2), \quad y^3 = \gamma \text{ in } F(q)$$

for every $f(i)$ of $F(i)$, and the quadratic field $F(q)$ is generated by q such that $q^2 = \xi$ in F. There exists an element k in $\mathfrak{Z}$ and not in $F(i)$ and an element y' in $\mathfrak{Z}$ such that

$$(23) \quad q(k) = -q(i), \quad (y')^3 = \gamma(-q), \quad (y')^r f(k) = f[\theta^r(k)](y')^r \qquad (r = 0, 1, 2),$$

so that if we write for any element

$$(24) \qquad X = a(i) + b(i)y + c(i)y^2, \quad X' = a(k) + b(k)y' + c(k)y'^2$$

then we have the properties

$$(25) \qquad \alpha' = \alpha, \quad (X + Y)' = X' + Y', \quad (XY)' = X'Y'$$

for every X and Y of $\mathfrak{Z}$ and α of F. Also there exists an element s in $\mathfrak{Z}$ such that if we write $X^{(\prime\prime\prime)} = (X')'$, $X^{(\prime\prime\prime\prime)} = (X'')', \cdots$, then

$$(26) \qquad s' = s, \quad X^{(\prime\prime\prime)} = sXs^{-1}.$$

Then there exists an element z in A such that the elements

$$(27) \qquad X_1 + X_2 z$$

where X_1 and X_2 range independently over all elements of $\mathfrak{Z}$ give all elements of A and A has the multiplication table

$$(28) \qquad zX = X'z, \quad z^2 = s.$$

Finally, the only elements of A commutative with q are elements of $\mathfrak{Z}$.

Conversely let i satisfy (20) and let there exist an element y such that the elements (21) and their multiplication table (22) are given and form an associative algebra $\mathfrak{A}$. Then $\mathfrak{A}$ is a division algebra if and only if $\gamma \neq a[\theta^2(x)]a[\theta(x)]a(x)$ for any polynomial $a(x)$. Let elements k, y' and s of $\mathfrak{A}$ be given so that (23) and (26) are satisfied. Then it is known that there exists an element z such that the elements (27) form the elements of an algebra A of type R_3 in 36 units with multiplication table (28). Also A is an associative normal algebra and is a division algebra if and only if $s \neq x'x$ for any x of $\mathfrak{A}$.

We shall now consider algebras B in thirty-six units of type R_6. Let B be such an algebra and let B contain x of type R_6.

LEMMA 4. *Let Ω be a substitution group of order six on six letters. Then Ω has a cyclic sub-group of order 3 and a cyclic sub-group of order 2.*

Proof. It is known * that the only groups of order and degree six are the cyclic group and one further group. This second group contains one sub-group of order three and three sub-groups of order two. The cyclic group may be written $\{I, \Theta, \Theta^2, \Theta^3, \Theta^4, \Theta^5\}$ where $\Theta^6 = I$. This contains the sub-group $\{I, \Theta^2, \Theta^4\}$ of order three and the sub-group $\{I, \Theta^3\}$ of order 2.

Hence if x has grade six and type R_6, since its minimum equation has a regular group, we may write the roots x, $\theta_1(x)$, $\theta_1^2(x)$, $\theta_2(x)$, $\theta_2[\theta_1(x)] = \theta_1[\theta_2(x)]$, $\theta_2[\theta_1^2(x)] = \theta_1^2[\theta_2(x)]$ such that $\theta_1^3(x) = \theta_2^2(x) = x$. Let first $\theta_2(x) + x = h \neq 0$. Let $g = x - \theta_2(x)$. If $g^2 = \xi$ in F, $h^2 = \eta$ in F, then either the field $F(g, h)$ has a basis 1, g, h, gh and hence has order 4 which is impossible since the grade of any element of B is 1, 2, 3, or 6, or $F(g, h) = F(g)$ of order two and h is in $F(g)$ and is unaltered by replacing $\theta_2(x)$ by x and x by $\theta_2(x)$, contrary to its form. Hence either ξ is not in F or η is not in F. Let $p = gh$. Then $-p$ is a root of the minimum equation of i since if we write $p = \theta(x)$ then if $\psi(p) = 0$ this equation is unaltered by replacing x by $\theta_2(x)$ and consequently g by $-g$. Hence the minimum equation of p has even powers only and p has grade two or six. If p has grade two then $p^2 = \zeta$ in F. But the minimum equation of g has even powers only so that either η is in F or g has grade six. But not both of η and ζ are in F since not both of ξ and η are in F and $\zeta = \xi\eta$. Therefore one of i and g has grade six and minimum equation $\psi(\omega^2) = 0$. Also if we call i the element g or p according as g or p has grade six then $F(i) = F(x)$ and $i = \theta(x)$.

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* See Netto, *Theory of Substitutions*, pp. 146-148.

Thus $\theta[\theta_1(x)]$, $\theta[\theta_1^2(x)]$ are roots of $\psi(\omega^2)=0$ and we obtain the following theorem

THEOREM 12. *Let x have grade 6 and type R_6. Then $F(x)$ contains an element i and a polynomial $\pi(i)$ such that the minimum equation of i is*

$$(29)\quad \psi(\omega)\equiv\omega^6+\lambda\omega^4+\mu\omega^2+\nu$$
$$\equiv[\omega+\pi^2(i)][\omega-\pi^2(i)][\omega+\pi(i)][\omega-\pi(i)][\omega+i][\omega-i]=0,$$
$$\lambda,\ \mu,\ \nu \text{ in } F.$$

It is obvious that i^2 has grade three and that $[\pi^2(i)]^2$, $[\pi(i)]^2$, i^2 are roots of its minimum equation commutative with it so that, by theorem 9,

THEOREM 13. *Let A be a normal division algebra in thirty-six units. Then A has type R_6 if and only if it contains an element of grade three commutative with one of its transforms not equal to itself.*

The algebras of type R_6 are those of the kind constructed by L. E. Dickson in his Transactions (1926) paper. These algebras are known algebras of type R_3 with the properties (22)-(26) where now, if we take i as the element of type R_6 and (29) as its minimum equation, we have $k=-i$. We have also found a necessary and sufficient condition that known algebras of type R_3 be of type R_6, in theorem 13.

PRINCETON, N. J.

On the r-th Divisors of a Number.

By D. H. Lehmer.

Introduction. Associated with every positive integer n is its set of distinct divisors (including 1 and n). Let us designate this set by δ_1. We have also a set of divisors associated with each number in the set δ_1. In this way the set δ_1 gives rise to a new set δ_2 of numbers (not all distinct) which are in fact the divisors of the divisors of n itself. If we take the divisors of the numbers in the set δ_2 we obtain a new set δ_3 and so on. The elements of the set δ_r we call the rth divisors of n. If $n > 1$ it is clear that the number of elements in the set δ_r increases with r. The order in which the elements in each set are arranged is of no consequence.

With the number n are also associated its proper divisors, those divisors of n which are less than n. This set of numbers we call d_1. Each number (> 1) in this set has one or more proper divisors. The set of all proper divisors of the numbers in the set d_1 we call d_2. The proper divisors of the numbers in the set d_2 form a set d_3 and so on. The numbers in the set d_r we call the rth proper divisors of n. The number of rth proper divisors of n ultimately becomes zero although it may increase for the first few values of r. It is convenient to think of sets δ_0 and d_0 each consisting of a single element, namely n itself.

The following example illustrates the nature of these sets of numbers in the case $n = 24$, and will be of use later.

$\delta_1 = 1, 2, 3, 4, 6, 8, 12, 24.$

$\delta_2 = 1 \quad 1,2 \quad 1,3 \quad 1,2,4 \quad 1,2,3,6 \quad 1,2,4,8 \quad 1,2,3,4,6,12$
$1,2,3,4,6,8,12,24.$

$\delta_3 = 1 \quad 1 \quad 1,2 \quad 1 \quad 1,3 \quad 1 \quad 1,2 \quad 1,2,4 \quad 1 \quad 1,2 \quad 1,3 \quad 1,2,3,6$
$1 \quad 1,2 \quad 1,2,4 \quad 1,2,4,8 \quad 1 \quad 1,2 \quad 1,3 \quad 1,2,4 \quad 1,2,3,6$
$1,2,3,4,6,12 \quad 1 \quad 1,2 \quad 1,3 \quad 1,2,4 \quad 1,2,3,6 \quad 1,2,4,8$
$1,2,3,4,6,12 \quad 1,2,3,4,6,8,12,24.$

. . . . . .

$d_1 = 1, 2, 3, 4, 6, 8, 12.$

$d_2 = 1 \quad 1 \quad 1,2 \quad 1,2,3 \quad 1,2,4 \quad 1,2,3,4,6.$

$d_3 = 1 \quad 1 \quad 1 \quad 1 \quad 1,2 \quad 1 \quad 1 \quad 1,2 \quad 1,2,3.$

$d_4 = 1 \quad 1 \quad 1 \quad 1$

For $r \geq 5$ the sets d_r are empty.

We first study the connection between the same arbitrary function of δ_r and of d_r which leads to certain inversion formulas. These we apply to the determination of the product and the sum of the kth powers of the rth divisors of n. Solutions of certain problems in the theory of probability can be obtained as a by-product of this investigation. If r is negative the definition of the rth divisors of n is meaningless. Nevertheless if we put negative values of r in the expressions for the above sums we obtain a sequence of functions different in nature but essentially belonging to the same class. Among these new functions we find Euler's totient function $\phi(n)$ and Merten's inversion function $\mu(n)$. In fact the whole investigation can be considered as a simultaneous extension of these two classical functions which thus appear to be closely connected.

Inversion of Numerical Integrals. The following investigation is simplified by the introduction of the concept of numerical integration. If two numerical functions $F(n)$ *and* $f(n)$ *are connected by the relation*

$$F(n) = \Sigma f(\delta_1) \tag{1}$$

where δ_1 ranges over the divisors of n, then $F(n)$ is said to be the numerical integral of $f(n)$; conversely $f(n)$ is the numerical derivative of $F(n)$. This concept is due to Bougaief * and has led to remarkable relations existing between the standard discontinuous functions of the theory of numbers. There are various inversion formulas by means of which we can express $f(n)$ in (1) in terms of $F(n)$. The most compact expression is perhaps

$$f(n) = \Sigma \mu(n/\delta_1) F(\delta_1) \tag{2}$$

which is due to Laguerre.† The function $\mu(n)$ is Merten's inversion function which is zero if n contains a square factor and $(-1)^t$ if n is a product of t distinct primes and is unity if $n = 1$. Another form of (2) is due to Dedekind:‡

$$f(n) = F(n) - \Sigma F(n/p_1) + \Sigma F(n/p_1p_2) - \Sigma F(n/p_1p_2p_3) + \cdots$$

where $n = p_1^{a_1}p_2^{a_2} \cdots p_t^{a_t}$. We assume throughout the article that n is expressed in this form. We also make use of the quantity

$$\omega = \alpha_1 + \alpha_2 + \cdots + \alpha_t,$$

which is called the order of n.

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* *Matematicheskii Sbornik*, Vol. 5 (1870-72), pp. 1-63; *Bulletin des Sciences Mathématiques et Astronomiques*, Vol. 10, pp. 13-32.

† *Bulletin de la Société Mathématique de France*, Vol. 1 (1872-3), pp. 77-81.

‡ *Journal für Mathematik*, Vol. 54 (1857), p. 21.

Introducing our sets d_r we can give another expression for $f(n)$ as follows:

$$(4) \quad f(n) = F(n) - \Sigma F(d_1) + \Sigma F(d_2) - \Sigma F(d_3) + \cdots (-1)^{\omega} \Sigma F(d_\omega).$$

To prove this formula we take for values of n the numbers in the set d_1 and n itself and sum. The result is

$$\Sigma f(d_1) + f(n) = \Sigma F(d_1) - \Sigma F(d_2) + \Sigma F(d_3) - \Sigma F(d_4) + \cdots$$
$$+ F(n) - \Sigma F(d_1) + \Sigma F(d_2) - \Sigma F(d_3) + \Sigma F(d_4) - \cdots.$$

Hence $\Sigma f(\delta_1) = F(n)$.

Formula (4) gives us new expressions for certain important functions. For example, if $\phi(n)$ is Euler's totient function we can write, in view of Gauss' theorem $n = \Sigma\phi(\delta_1)$, that

$$\phi(n) = n - \Sigma(d_1) + \Sigma(d_2) - \Sigma(d_3) + \cdots.$$

THEOREM 1. *Let $F(n)$ be an arbitrary function of n then for every integer r we have*

$$(5) \quad (-1)^r \Sigma F(d_r) = F(n) - \binom{r}{1}\Sigma F(\delta_1) + \binom{r}{2}\Sigma F(\delta_2) - \cdots (-1)^r \Sigma F(\delta_r).$$

In fact $\Sigma F(d_1) = \Sigma F(\delta_1) - F(n)$.

To obtain $\Sigma F(d_2)$ we sum over the proper divisors of δ_1 and n, namely over the set $\delta_2 - \delta_1$ and the set $\delta_1 - n$. That is,

$$\Sigma F(d_2) = \Sigma F(\delta_2) - 2\Sigma F(\delta_1) + F(n).$$

Similarly

$$\Sigma F(d_3) = \Sigma F(\delta_3) - 3\Sigma F(\delta_2) + 3\Sigma F(\delta_1) - F(n).$$

From the relation

$$(6) \quad \binom{m-1}{l-1} + \binom{m-1}{l} = \binom{m}{l}$$

we have by a simple induction

$$\Sigma F(d_r) = \Sigma F(\delta_r) - \binom{r}{1}\Sigma F(\delta_{r-1}) + \binom{r}{2}\Sigma F(\delta_{r-2}) - \cdots + (-1)^r F(n),$$

which is the theorem. In the same way it can be shown that

$$(7) \quad \Sigma F(\delta_r) = \Sigma F(d_r) + \binom{r}{1}\Sigma F(d_{r-1}) + \binom{r}{2}\Sigma F(d_{r-2}) + \cdots + F(n).$$

If we sum equations of the type (5) for $r = 0, 1, 2, \cdots, \omega$ we obtain in view of (4)

$$(8) \quad f(n) = \binom{\omega+1}{1} F(n) - \binom{\omega+1}{2} \Sigma F(\delta_1) + \binom{\omega+1}{3}\Sigma(\delta_2) - \cdots + (-1)^{\omega}\Sigma F(\delta\omega),$$

which gives us a new inversion formula. As a matter of fact, if the index r in (5) exceeds ω, then $\Sigma F(d_r) = 0$. So that in (8) we may take in place of ω any larger number and the result will be the same. As an example of (8) we may put $f(n) = n$ and get another expression for Euler's totient function in the form

$$(9) \qquad \phi(n) = \binom{r}{1}n - \binom{r}{2}\Sigma\delta_1 + \binom{r}{3}\Sigma\delta_2 - \cdots + (-1)^{r-1}\Sigma\delta_{r-1},$$

where $r \geq \omega + 1$.

For instance, if $n = 6$ we have

$$\delta_1 = 1, 2, 3, 6.$$
$$\delta_2 = 1, \quad 1, 2, \quad 1, 3, \quad 1, 2, 3, 6.$$
$$\delta_3 = 1, \quad 1, \quad 1, 2, \quad 1, \quad 1, 3, \quad 1, \quad 1, 2, \quad 1, 3, \quad 1, 2, 3, 6.$$
$$\delta_4 = 1, \quad 1, \quad 1, \quad 1, 2, \quad 1, \quad 1, \quad 1, 3, \quad 1, \quad 1, \quad 1, 2, \quad 1, \quad 1, 3, \quad 1, \quad 1, 2,$$
$$1, 3, \quad 1, 2, 3, 6.$$
$$\Sigma\delta_1 = 12, \quad \Sigma\delta_2 = 20, \quad \Sigma\delta_3 = 30, \quad \Sigma\delta_4 = 42.$$

Applying (9) we have for $r = 3, 4, 5$,

$$\phi(6) = 3.6 - 3.12 + 20 = 2,$$
$$\phi(6) = 4.6 - 6.12 + 4.20 - 30 = 2,$$
$$\phi(6) = 5.6 - 10.12 + 10.20 - 5.30 + 42 = 2.$$

If F and f are related not as in (1) but as follows

$$(1') \qquad F(n) = \Pi f(\delta_1),$$

or what is the same

$$\log F(n) = \Sigma \log f(\delta_1),$$

the above inversion formulas (4) and (8) go over into

$$(4') \qquad f(n) = \frac{F(n) \cdot \Pi F(d_2) \cdot \Pi F(d_4) \cdots}{\Pi F(d_1) \cdot \Pi F(d_3) \cdots}$$

$$(8') \qquad f(n) = \frac{[F(n)]^{\binom{\omega+1}{1}} \cdot [\Pi F(\delta_2)]^{\binom{\omega+1}{3}} \cdots}{[\Pi F(\delta_1)]^{\binom{\omega+1}{2}} \cdot [\Pi F(\delta_3)]^{\binom{\omega+1}{4}} \cdots}$$

Successive Numerical Integrals. Let us consider a sequence of functions

$$(10) \qquad \cdots f_{-3}, \quad f_{-2}, \quad f_{-1}, \quad f_0, \quad f_1, \quad f_2, \quad f_3, \cdots$$

each of which is the numerical integral of its predecessor. That is

$$(11) \qquad f_r(n) = \Sigma f_{r-1}(\delta_1).$$

If we define one of the functions, say $f_0(n)$, for all values of n, then every function $f_r(n)$ is completely determined.

These functions f_r have certain properties in common. For instance

$$f_r(1) = f_0(1) \qquad \text{for every } r. \tag{12}$$

The following multiplicative property is important for our purposes:

$$f(m) \cdot f(n) = f(mn), \tag{13}$$

where m and n are relatively prime integers.

THEOREM 2. *If any function f_r of the set* (10) *possesses the multiplicative property* (13), *so does every function of the set.*

The proof is by induction. We shall first show that f_{r+1} has the property. From (11) we have

$$f_{r+1}(n) \cdot f_{r+1}(m) = \sum_{\delta_1,\delta_1'} f_r(\delta_1) \cdot f_r(\delta_1'), \tag{14}$$

where δ_1' ranges over the divisors of m. Since m and n are prime to each other and since (13) holds for f_r, the right side of (14) becomes $\Sigma f_r(\delta_1\delta_1')$. As δ_1 and δ_1' run over their respective values their product ranges over the divisors of mn without repetition. Hence

$$f_{r+1}(n) \cdot f_{r+1}(m) = f_{r+1}(mn).$$

To prove that (13) holds for f_{r-1} we make use of (2) and write

$$\begin{aligned} f_{r-1}(n) \cdot f_{r-1}(m) &= \sum_{\delta_1,\delta_1'} \mu(n/\delta_1) f_r(\delta_1) \cdot \mu(n/\delta_1') f_r(\delta_1') \\ &= \Sigma\, \mu(mn/\delta_1\delta_1') f_r(\delta_1\delta_1'), \end{aligned}$$

since $\mu(n)$ has the multiplicative property.* Thus

$$f_{r-1}(n) \cdot f_{r-1}(m) = f_{r-1}(mn).$$

Hence the theorem is proved.

The expression (11) can be generalised at once to give the relation between f_r and f_{r+s}.

$$\begin{aligned} f_{r+1}(n) &= \Sigma f_r(\delta_1), \\ f_{r+2}(n) &= \Sigma f_{r+1}(\delta_1) = \Sigma f_r(\delta_2), \\ &\cdots\cdots\cdots\cdots \\ f_{r+s}(n) &= \Sigma f_r(\delta_s) = \Sigma f_s(\delta_r). \end{aligned} \tag{15}$$

Comparing (15) and (7) we have

$$f_{r+s}(n) = f_r(n) + \binom{s}{1} \Sigma f_r(d_1) + \binom{s}{2} \Sigma f_r(d_2) + \cdots + \Sigma f_r(d_s). \tag{16}$$

This expression for the integral of order s of f_r is to be preferred to (15) for computational purposes. After the finite number of sums

* This simple fact follows at once from the definition of $\mu(n)$.

$$\Sigma f_r(d_i) \qquad (i = 0, 1, 2, 3, \cdots, \omega)$$

have been calculated, the integral of any order may be obtained as the linear combination of these with the proper binomial coefficients.

Example. Let $f_r(n) = P_0(n)$, the number of primes less than or equal to n. The problem of getting an explicit formula even for the first integral is a very difficult one. Nevertheless we can easily calculate the numerical value of the integral of any order for a given n. For instance for $n = 24$ referring to the sets d_r already given in the introduction, we have the following values for the sums:

$$P_0(n) = 9, \quad \Sigma P_0(d_1) = 17, \quad \Sigma P_0(d_2) = 15, \quad \Sigma P_0(d_3) = 5, \quad \Sigma P_0(d_4) = 0.$$

Hence

$$P_s(24) = 9 + 17\binom{s}{1} + 15\binom{s}{2} + 5\binom{s}{3}.$$

We can express $f_r(n)$ in (15) in terms of $f_{r+s}(n)$ by making successive applications of (2) with the result

$$f_r(n) = \Sigma \mu(n/\delta_s) f_{r+s}(\delta_s), \tag{17}$$

which gives us a way of calculating isolated derivatives of a given function f_{r+s}. Successive applications of the formula (4) give an expression similar to (16), namely

$$f_r(n) = f_{r+s}(n) - \binom{s}{1}\Sigma f_{r+s}(d_1) + \binom{s+1}{2}\Sigma f_{r+s}(d_2) - \cdots + (-1)^{\omega}\binom{\omega+s-1}{\omega}\Sigma f_{r+s}(d_\omega). \tag{18}$$

Putting $n = 24$ and $s = 5$ we calculate $P_5(\delta_1)$ from (16) with the result

$$P_5(24) = 294, \quad P_5(12) = 85, \quad P_5(8) = 29, \quad P_5(6) = 18,$$
$$P_5(4) = 7, \quad P_5(3) = 2, \quad P_5(2) = 1.$$

Using these values in (18) we have

$$P_0(24) = 294 - \binom{5}{1}\cdot 145 + \binom{6}{2}\cdot 40 - \binom{7}{3}\cdot 5 = 9,$$

which is, in fact, the number of primes less than 24.

In view of relation (15) we can transform (5) to read

$$(-1)^s\Sigma f_r(d_s) = f_r(n) - \binom{s}{1}f_{r+1}(n) + \binom{s}{2}f_{r+2}(n) - \cdots (-1)^s f_{r+s}(n) = \sum_{i=0}^{s}(-1)^i\binom{s}{i}f_{r+i}(n). \tag{19}$$

This last sum is zero for all values of $s > \omega$ regardless of what sequence of functions f_r we select.

In the same way the inversion formula (8) may be written

$$f_r(n) = \binom{h}{1} f_{r+1}(n) + \binom{h}{2} f_{r+2}(n) + \binom{h}{3} f_{r+3}(n) - \cdots + (-1)^{h-1} f_{r+h}(n), \tag{20}$$

where $h > \omega$.

The Function $\sigma_{k,r}(n)$. We shall consider as a special case of (10) the sequence $\sigma_{k,r}(n)$ determined by putting $f_0(n) = n^k$. From (15) it follows that

$$\sigma_{k,r}(n) = \Sigma \sigma_{k,0}(\delta_r) = \Sigma (\delta_r)^k,$$

which shows that $\sigma_{k,r}(n)$ is the sum of the k-th powers of the r-th divisors of n.

THEOREM 3. *If p is a prime, then*

$$\sigma_{k,r}(p^a) = p^{ka} + \binom{r}{1} p^{k(a-1)} + \binom{r+1}{2} p^{k(a-2)} + \cdots = \sum_{j=0}^{a} \binom{j+r-1}{j} p^{k(a-j)}.$$

The theorem is evidently true for $r = 0$. The proof of the theorem consists in showing that if it is true for the index r, it is also true for both the indices $r-1$ and $r+1$, the first part of the proof being required for the discussion of negative indices. First we may write

$$\sigma_{k,r}(p^a) = \sigma_{k,r-1}(p^a) + \sum_{j=0}^{a-1} \sigma_{k,r-1}(p^j), \tag{21}$$

$$\sigma_{k,r-1}(p^a) = \sigma_{k,r}(p^a) - \sigma_{k,r}(p^{a-1}).$$

Supposing the theorem to be true for the index r we have

$$\sigma_{k,r-1}(p^a) = p^{ka} + \binom{r}{1} p^{k(a-1)} + \binom{r+1}{2} p^{k(a-2)} + \binom{r+2}{3} p^{k(a-3)} + \cdots - p^{k(a-1)} - \binom{r}{1} p^{k(a-2)} - \binom{r+1}{2} p^{k(a-3)} - \cdots.$$

In view of (6)

$$\sigma_{k,r-1}(p^a) = p^{ka} + \binom{r-1}{1} p^{k(a-1)} + \binom{r}{2} p^{k(a-2)} + \binom{r+1}{3} p^{k(a-3)} + \cdots.$$

Hence the theorem holds true for the index $r-1$.

To prove that the theorem is true for $r+1$ we use an induction on the exponent α. Writing (21) in the form

$$\sigma_{k,r+1}(p^a) = \sigma_{k,r}(p^a) + \sigma_{k,r+1}(p^{a-1}), \tag{22}$$

and putting $\alpha = 1$ we have

$$\sigma_{k,r+1}(p) = \sigma_{k,r}(p) + \sigma_{k,r+1}(1).$$

Using property (12) and the hypothesis of the induction we have

$$\sigma_{k,r+1}(p) = p^k + \binom{r}{1} + 1 = p^k + \binom{r+1}{1}.$$

Hence the theorem is true for $\sigma_{k,r+1}(p)$. To complete the proof we must show that if the theorem holds for $\sigma_{k,r+1}(p^{a-1})$ and $\sigma_{k,r}(p^a)$ it is true for $\sigma_{k,r+1}(p^a)$. With these hypotheses equation (22) becomes

$$\begin{aligned}\sigma_{k,r+1}(p^a) &= p^{ka} + \binom{r}{1}p^{k(a-1)} + \binom{r+1}{2}p^{k(a-2)} + \cdots \\ &\quad + p^{k(a-1)} + \binom{r+1}{1}p^{k(a-2)} + \cdots \\ &= p^{ka} + \binom{r+1}{1}p^{k(a-1)} + \binom{r+2}{2}p^{k(a-2)} + \cdots.\end{aligned}$$

This establishes the theorem for the index $r+1$ and hence for all indices.

It is obvious that $\sigma_{k,0}(n)$ has the multiplicative property (13). By Theorem 2 all σ's have this property. This leads us at once to the general expression for $\sigma_{k,r}(n)$ namely

$$\sigma_{k,r}(n) = \prod_{i=1}^{t}\sum_{j=0}^{a_i}\binom{j+r-1}{j}p_i^{k(a_i-j)}. \tag{23}$$

The function $\mu_r(n)$. We call $\mu_r(n)$ the special case of the function $\sigma_{k,r}(n)$ for which $k=0$. From the last equation,

$$\mu_r(n) = \prod_{i=1}^{t}\sum_{j=0}^{a_i}\binom{j+r-1}{j} = \prod_{i=1}^{t}\binom{r+a_i}{r}. \tag{24}$$

The function $\mu_r(n)$ gives the number of r-th divisors of n, that is the number of elements in the set δ_r. For $r=1$, $\mu_1(n) = \prod_{i=1}^{t}(a_i+1)$, which the well known formula for the number of first divisors of n. The function $\mu_r(n)$ does not depend on the actual prime factors of n but upon their number and multiplicity. In case n is a product of t distinct primes $\mu_r(n) = (r+1)^t$.

Example. Let $n = p_1^3p_2^2p_3$, then $\mu_r(n) = (r+1)^3(r+2)^2(r+3)/12$, which gives for the first five values of r:

$$\begin{aligned}&\mu_0(n)=1, && \mu_1(n)=24, && \mu_2(n)=180,\\ &\mu_3(n)=800, && \mu_4(n)=2625, && \mu_5(n)=7056.\end{aligned}$$

THEOREM 4. *The number of r-th proper divisors of n is given by*

$$M_r(n) = \mu_r(n) - \binom{r}{1}\mu_{r-1}(n) + \binom{r}{2}\mu_{r-2}(n) - \cdots + (-1)^r\mu_0\cdot(n). \tag{25}$$

This follows at once from equation (19). If n is a product of t distinct primes, (25) becomes

$$M_r(n) = (r+1)^t - \binom{r}{1}r^t + \binom{r}{2}(r-1)^t - \binom{r}{3}(r-2)^t + \cdots + (-1)^r.$$

If $r > t$ this expression is zero since the sets d_r are empty. The vanishing

of this expression for $r > t$ has been noted in connection with binomial series.*

If we introduce negative values of the index, the function $\mu_r(n)$ becomes †

$$(26)\quad \mu_{-r}(n) = \prod_{i=1}^{t} \binom{a_i - r}{a_i} = \prod_{i=1}^{t} (-1)^{a_i} \binom{r-1}{a_i} = (-1)^{\omega} \prod_{i=1}^{t} \binom{r-1}{a_i}.$$

The function $\mu_{-1}(n)$ is zero if $n > 1$ and unity if $n = 1$. The function $\mu_{-2}(n)$ is zero if n contains a square factor, is $(-1)^t$ if n is a product of t distinct primes and is unity if $n = 1$. In short $\mu_{-2}(n)$ is Merten's inversion function. This proves that if a function $f(n)$ is such that $\Sigma f(\delta_1)$ is zero for $n > 1$ and is unity for $n = 1$, then $f = \mu$.‡ The function $\mu_{-3}(n)$ vanishes if n is divisible by a cube. Otherwise its value is $(-2)^{\lambda}$ where λ is the number of prime factors of n which appear to the first power only. In general $\mu_{-r}(n)$ is zero if n is divisible by a perfect r-th power.

Example. If we take again $n = p_1^3 p_2^2 p_3$, we have

$$\mu_{-1} = \mu_{-2} = \mu_{-3} = 0, \quad \mu_{-4} = 9, \quad \mu_{-5} = 96, \quad \mu_{-6} = 500, \quad \mu_{-7} = 1800.$$

The function $\phi_r(n)$. We define the function

$$(27)\qquad \phi_r(n) \equiv \sigma_{1,r}(n) = \prod_{i=1}^{t} \sum_{j=0}^{a_i} \binom{j+r-1}{r-1} p_i^{a_i - j}$$

which gives the sum of the r-th divisors of n. If $r = 1$ we have

$$\phi_1(n) = \prod_{i=1}^{t} (p_i^{a_i} + p_i^{a_i - 1} + \cdots + 1) = \prod_{i=1}^{t} (p_i^{a_i+1} - 1)/(p_i - 1),$$

which is well known. For $r = 2$ we have

$$\phi_2(n) = \prod_{i=1}^{t} [p_i^{a_i+2} - p_i(\alpha_i + 2) + \alpha_i + 1]/(p_i - 1)^2,$$

which gives the sum of the numbers in the set δ_2. The function

$$\phi_3(n) = \prod_{i=1}^{t} [p_i^{a_i+3} - \binom{a_i+3}{2} p_i^2 + (\alpha_i + 1)(\alpha_i + 3) p_i - \binom{a_i+2}{2}]/(p_i - 1)^3,$$

gives the sum of the third divisors of n, and so on.

For negative indices, (27) becomes §

$$\phi_{-r} = \prod_{i=1}^{t} \sum_{j=0}^{a_i} (-1)^j p^{a_i - j} \binom{r}{j}.$$

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* See *Chrystal's Algebra*, Vol. 2, p. 183 (3).

† Adopting the usual definition $\binom{-m}{l} = (-1)^l \binom{m+l-1}{l}$ so that (6) holds.

‡ This is the converse of Merten's theorem. *Journal für Mathematik*, Vol. 78 (1874), p. 53.

§ N. V. Bervi, *Matematicheskii Sbornik*, Vol. 18, pp. 525-8.

If $r=1$ we have

$$\phi_{-1}=\prod_{i=1}^{t}(p^{a_i}-p^{a_i-1}),$$

which is Euler's totient function.* This converse of Gauss' theorem has been given by Lucas.† The functions ϕ_{-r} for $r>1$ have certain similarities to Euler's $\phi(n)$. For instance

$$\begin{aligned}&\phi_{-r}(p^a)=p^{a-r}(p-1)^r \qquad a\geqq r\\ &\phi_{-r}(p)=p-r.\end{aligned} \tag{28}$$

THEOREM 5. *If n contains the prime factor p to the first but no higher power, then $\phi_{-p}(n)=0$.*

This follows at once from (28).

The following is a list of values of $\phi_r(n)$ for $n=42$ and $n=56$

$r=$	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3
$\phi_r(42)=$	−168	−84	−30	0	12	12	6	0	0	12	42	96	180	300
$\phi_r(56)=$	186	80	24	0	−6	−4	0	4	10	24	56	120	234	420

The function $\pi_r(n)$. In conclusion we shall develop a formula for the product $\pi_r(n)$ of the r-th divisors of n. We shall use the following lemma

LEMMA. The number of times that a certain divisor Δ of n appears in the set δ_r is $\mu_{r-1}(n/\Delta)$.

Evidently the lemma is true for $r=1$ since δ_1 consists of distinct elements and $\mu_0=1$. Let us assume that the lemma is true for the set δ_{r-1}. If we form the set δ_r from the divisors of the set δ_{r-1} the element Δ can arise only from numbers of the form $\Delta\delta'$ where δ' are the divisors of (n/Δ) and from each of these it can arise but once. By the hypothesis of our induction the number of times each divisor of the form $\Delta\delta'$ appears in δ_{r-1} is $\mu_{r-2}(n/\Delta\delta')$. The total number of Δ's appearing in δ_r is therefore

$$\Sigma\mu_{r-2}(n/\Delta\delta')=\Sigma\mu_{r-2}(\delta')=\mu_{r-1}(n/\Delta)$$

since δ' ranges over the divisors of (n/Δ). Thus the lemma is true for all positive values of r.

Let $f_s(n)$ be any function of an arbitrary sequence. Then from the lemma it follows that

$$f_{r+s}(n)=\Sigma f_s(\delta_r)=\Sigma\mu_{r-1}(n/\delta_1)f_s(\delta_1). \tag{29}$$

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* In general $\sigma_{k,-1}(n)=n^k\prod_{i=1}^{t}(1-P_i^{-k})$. This function has been the subject of a number of investigations. See D. N. Lehmer, *American Journal of Mathematics*, Vol. 22 (1900), pp. 293-335.

† *Théorie des Nombres*, p. 401.

This gives another short method of calculating the actual values of isolated integrals and derivatives. In this method it is unnecessary to determine the numbers in the sets δ_k for $k > 1$. Thus to find $P_5(n)$ in Example 1 we put $f_s(n) = P_0(n)$ and $r = 5$. The results are as follows

$$\begin{aligned}
\delta_1 &= 1, \quad 2, \quad 3, \quad 4, \quad 6, \quad 8, \quad 12, \quad 24, \\
P_0(\delta_1) &= 0, \quad 1, \quad 2, \quad 2, \quad 3, \quad 4, \quad 5, \quad 9, \\
\mu_4(n/\delta_1) &= 175, \quad 75, \quad 35, \quad 25, \quad 15, \quad 5, \quad 5, \quad 1, \\
P_5(n) &= 0 + 75 + 70 + 50 + 45 + 20 + 25 + 9 = 294,
\end{aligned}$$

which agrees with our previous result.

In (29) r and s are interchangable and since we chose s arbitrarily r is similarly at our disposal that is (29) will hold for $r \leq 0$, although the expression $\Sigma f_s(d_r)$ fails to have any meaning in this case. If we put $r = -1$ we have

$$f_{s-1}(n) = \Sigma \mu_{-2}(n/\delta_1) f_s(\delta_1).$$

Since μ_{-2} is Merten's function this expression is simply Leguerre's inversion formula (2). From this standpoint μ_{-r} may be thought of as an inversion function of order $r-1$. Putting $f_s(n) = n$ in (29) we have the following compact expression for $\phi_r(n)$

$$\phi_r(n) = \Sigma \delta_1 \mu_{r-1}(n/\delta_1).$$

If we wish to find the number of times a divisor Δ appears in the set d_r we use equation (5) with $F(n) = 0$ for n different from Δ and $F(\Delta) = 1$. This gives the result

$$\mu_{r-1}(n/\Delta) - \binom{r}{1}\mu_{r-2}(n/\Delta) + \binom{r}{2}\mu_{r-3}(n/\Delta) - \cdots + (-1)^r \mu_{-1}(n/\Delta).$$

THEOREM 6. *The product $\pi_r(n)$ of the r-th divisors of n is $n^{\mu_r(n)/(r+1)}$.* Let us select a certain prime factor p of n and write $n = p^a m$. We shall determine the power to which p appears in the desired product. Let us first consider the numbers of the set δ_r which contain p to the first but not to any higher power. Such numbers are of the form pm' where m' are the divisors of m. By our lemma the number of such elements is

$$\Sigma \mu_{r-1}(p^{a-1}m/m') = \mu_{r-1}(p^{a-1}) \Sigma \mu_{r-1}(m') = \mu_{r-1}(p^{a-1}) \mu_r(m).$$

This is the contribution to the power of p from the set of elements pm'. The corresponding contributions from the numbers of the form pm' are

$$j\mu_r(m)\mu_{r-1}(p^{a-j}) \qquad (j = 1, 2, 3 \cdots \alpha).$$

Summing these contributions we have

$$\mu_r(m)\sum_{j=1}^{a} j\mu_{r-1}(p^{a-j}) = \mu_r(m)[\mu_r(p^{a-1}) + \mu_r(p^{a-2}) + \cdots + 1]$$
$$= \mu_r(m)\,\mu_{r+1}(p^{a-1}).$$

Since $$\mu_{r+1}(p^{a-1}) = \mu_r(p^a)\cdot a/(r+1)$$

the power to which p appears in the final product is

$$[a/(r+1)]\mu_r(m)\,\mu_r(p^a) = [a/(r+1)]\mu_r(n).$$

This result being true for all primes dividing n we have

$$\pi_r(n) = \prod_{i=1}^{t} p_i^{a_i\mu_r(n)/(r+1)} = n^{\mu_r(n)/(r+1)},$$

which is the theorem.

For $r = 1$ we have $\pi_1(n) = n^{\mu_1(n)/2}$. This well known result follows at once by associating with every divisor $\delta_1 \leqq n^{1/2}$ its co-divisor. This method fails however for $r > 1$.

The product of the numbers in the set d_r may be readily obtained from inversion formula (5) putting $\Sigma F(\delta_k) = [\mu_k(n)/(k+1)]\log n$.

Corollary. The geometric mean of the numbers in the set δ_r is $n^{1/(r+1)}$. This follows immediately from the theorem since $\mu_r(n)$ gives the number of elements in the set δ_r. It is interesting to notice that the mean can be calculated without reference to the prime factors of n and two numbers such as 8191 and 8192 have practically the same mean although their compositions are widely different.

Although the foregoing discussion was carried out in number-theoretic terminology it is of much wider application. We might consider a collection of ω things of which α_1 are of one sort, α_2 of a second sort and so on. It is well known that the number of selections that can be made from such a collection is the same as the number of first divisors of $n = p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$. The function $\mu_r(n)$ has immediate application to a similar problem in which r successive selections are made. If we count as inadmissible the selection in which all things are rejected, the r-th proper divisors of n must be considered. If we assign different weights to the different sorts of things the function $\phi_r(n)$ becomes essential to the discussion. In a future paper we hope to apply the theory of r-th divisors to certain problems in combinatory analysis.

On Harmonics Applicable to Surfaces of Revolution.

By D. M. Wrinch.

Introduction.

1. A large class of problems in mathematical physics may be described in the following manner. A function V is required, satisfying the equation of Laplace, which is evanescent on S, the sphere at infinity, has no singularities in the region between S and a certain closed surface s, and on this surface s takes a certain specific form. This general class includes problems of electrical distributions on conductors, both when the conductor is freely charged and when its electrification is due to an external field of electric force. It further covers problems of hydrodynamics, when, for example, there is streaming past an obstacle, or when a body has uniform translational or rotational motion in an infinite liquid.

2. Problems of this type, though worked out with some degree of completeness for two dimensional surfaces,* have been solved for but a few three dimensional surfaces. Those associated with the sphere can be solved in terms of the spherical harmonics

$$r^{-k-1} P_k^m(\cos\theta) \cdot [\cos m\omega, \sin m\omega]$$

where r, θ, ω are the usual spherical polar coördinates measured from the centre of the sphere as origin. In addition to the case of the prolate and oblate spheroids, these problems have also been fully worked out for inverted oblate spheroids, the centre of inversion being on the axis of figure and they may also, evidently, be discussed for prolate spheroids, similarly inverted.† These extensions of the technique for dealing with spheroids proved possible owing to the fact that if harmonics are available for any surface they are also available for the inverse surface. Ellipsoidal harmonics, further, make the treatment of problems associated with ellipsoids theoretically possible,

* See a series of papers by D. M. Wrinch: *Philosophical Magazine*, Vol. 48 (1924), pp. 692-703 and pp. 1089-1104; also Vol. 49 (1925), pp. 240-250; *Journal of the Royal Aeronautical Society*, Vol. 30 (1926), pp. 129-141.

† See papers by D. M. Wrinch in the *Philosophical Magazine*, Vol. 53 (1927), pp. 865-883; and Vol. 50 (1925), pp. 1049-1058.

though the practical difficulties are so great that very little progress has, in fact, been made.

3. It appears, however, that there has been very little methodological treatment of these problems, whereby they may be attacked even when the surface is of a general type. It seemed worth while, therefore, to put forward a method of constructing harmonics which will allow treatment of problems in which there is symmetry about an axis, when associated with a surface of revolution whose equation is

$$(1)\quad z = z(u) = \alpha \cos u + \alpha_1 \cos u + \alpha_2 \cos 2u + \cdots + \alpha_n \cos nu + \cdots$$
$$\rho = \rho(u) = \alpha \sin u - \alpha_1 \sin u - \alpha_2 \sin 2u - \cdots - \alpha_n \sin nu - \cdots$$

(the axis of symmetry being the axis of figure), provided that the coefficients α, α_n satisfy certain general conditions. Thus, in the first place, we shall assume that the transformation

$$(2)\qquad W = W(w) = \alpha e^{iw} + \alpha_1 e^{-iw} + \alpha_2 e^{-2iw} + \cdots + \alpha_n e^{-niw} + \cdots$$

where $w = u + iv$ is such that dW/dz is never zero and never infinite for values of v between zero and $-\infty$. The zero v level evidently represents the surface s, and the v level given by $v = -\infty$ the sphere at infinity: the assumption then ensures that dW/dz has no zeros and no infinities in the region between s and S, the sphere at infinity. A further condition relating to the orders of magnitude of the coefficients will also be introduced in due course.

4. The harmonics to be constructed are interesting in that they are all simply series of Legendre Functions $P_k(\cos u)$ multiplied by polynomials in e^v. This fact renders the method of some practical importance, since the Legendre functions are already adequately tabulated and easy to handle. It also has the consequence that much of the theory of Legendre functions already worked out will undoubtedly prove useful and illuminating in the treatment of surfaces which possess a more general equation than the sphere and spheroid previously treated.

5. But the chief interest of the method is of a different kind. In the case of the sphere, given by the zero v level in the transformation $W = \alpha e^{iw}$, problems involving solutions symmetric about the z axis are solved in terms of the harmonics

$$P_k(\cos u)\ e^{(k+1)v}$$

If we go one step on the road towards generality and treat the spheroid by means of the transformation

$$W = \alpha e^{iw} + \alpha_1 e^{-iw}, \tag{3}$$

(in which $\alpha_1^2 < \alpha^2$) there is a sense in which the harmonics for the sphere are 'corrected' into harmonics for the spheroid, of type

$$P_k(\cos u)\ F_k(e^v),$$

although the customary treatment by means of spheroidal harmonics has obscured the fact.

Now there is also a sense in which harmonics available for any surface of the type given in (1) are 'corrected' forms of spheroidal harmonics, but this procedure has the disadvantage that the standard forms for the harmonics would then involve the Q_k and q_k Legendre functions, which do not lend themselves readily to simple computation. We prefer therefore to take the spherical harmonics as the starting point and to build up harmonics for more general surfaces from them.

The advantage of this point of view is that it enables us to look at the problems as a whole, whatever type of surface of revolution is under discussion, provided only that it falls under the equation for some set of values or other of the α_n parameters which they are free to take. The way to obtain harmonics for any special surface is theoretically plain and in any particular case presents no mathematical difficulties of any sort. Indeed, it is not difficult to elaborate a technique whereby the form of harmonic suitable for a surface s can be 'corrected' as we pass from the spherical surface treated by $W = \alpha e^{iw}$ through intermediate stage to the greater generality covered by the transformation (2) in its various cases. The interesting point is that there is an inductive procedure always available for these problems whereby a more general form for the surface s does not require a wholly new treatment *ab initio*, but merely the correction of the harmonics in certain clearly defined ways. It is not necessary to wait for a kind of mathematical mutation before a new type of surface can be treated.

Our objective, therefore, is to put forward a method which is, mathematically speaking, of the utmost simplicity, by means of which we can construct harmonics symmetric about the axis of z in such a form that those suitable for any surface covered by the equations (1) can, in due course, be investigated. It is an aesthetic misfortune that although the structure of the harmonics is simple enough and entirely elegant, yet the coefficients

involved are arithmetically complicated. Since it is no part of our plan in this paper to work out pages of arithmetical calculations, having worked out a few particular cases and having explained the method and its scope, we shall leave on one side the laborious determination of the coefficients involved in other still more general cases. It may, perhaps, turn out to be the case that new results relating to Legendre functions will be required before any complete account of them can be given.

6. It will be noticed that there is a considerable advantage from the point of view of applications in the fact that a step by step method of building harmonics is available. In practice it is of the first importance that results should be expressed in such a form that an approximation to any required degree is immediately deducible. Indeed it is sometimes the case that elegant solutions of problems in pure mathematics give only cold comfort to those who wish to use them to obtain further information about the external world. We therefore think it important to draw attention to the fact that the harmonics developed in this paper are of a type which allows the solution of problems of physics to be pursued to just that degree of approximation which the particular circumstances of each problem may require.

I. *The appropriate form for Laplace's Equation.*

1.1. When

$$x,\ y = \rho(\cos\omega,\ \sin\omega)$$
$$z + i\rho = W(w) = W(u + iv)$$

we take the line elements normal to the u, v, ω levels in the form

$$dn_u = du/h_u,\ dn_v = dv/h_v,\ dn_\omega = d\omega/h_\omega$$

and in the usual manner

$$1/h^2_{u,v,\omega} = (\partial x/\partial u,\ v,\ \omega)^2 + (\partial y/\partial u,\ v,\ \omega)^2 + (\partial z/\partial u,\ v,\ \omega)^2$$

so that

$$1/h^2_\omega = \rho^2$$
$$1/h^2_u = 1/h^2_v = |\ dw/dz\ |^2 = 1/h^2$$

In consequence, Laplace's equation takes the form

$$\nabla^2 V \equiv h^2\left[\frac{1}{\rho}\ \frac{d}{du}\ \rho\ \frac{d}{du} + \frac{1}{\rho}\ \frac{d}{dv}\ \rho\ \frac{d}{dv} + \frac{1}{\rho^2 h^2}\ \frac{d^2}{d\omega^2}\right] V = 0,$$

and solutions symmetrical about the axis of z, being independent of ω, therefore have the determining equation

$$\left[\frac{d^2}{du^2} + \frac{1}{\rho}\frac{d\rho}{du}\frac{d}{du} + \frac{d^2}{dv^2} + \frac{1}{\rho}\frac{d\rho}{dv}\frac{d}{dv}\right] V = 0.$$

Now with

$$Z + i\rho = W(u+iv) = \alpha e^{iw} + \alpha_1 e^{-iw} + \alpha_2 e^{-2iw} + \cdots + \alpha_n e^{-niw} + \cdots$$

as in (2), the equation takes the form

$$(4) \qquad \phi(u,v)[(d^2/du^2) + (d^2/dv^2)]V + \psi(u,v)(dV/du) + \chi(u,v)(dV/dv) = 0$$

where

$$\phi \equiv (\alpha e^{-v} - \alpha_1 e^{v}) \sin u - \alpha_2 e^{2v} \sin 2u \cdots$$
$$\psi \equiv (\alpha e^{-v} - \alpha_1 e^{v}) \cos u - 2\alpha_2 e^{2v} \cos 2u \cdots$$
$$-\chi \equiv (\alpha e^{-v} + \alpha_1 e^{v}) \sin u + 2\alpha_2 e^{2v} \sin 2u \cdots$$

which may be written in terms of operators D, D_n

$$(5) \qquad (\alpha D - \alpha_1 D_1 - \alpha_2 D_2 \cdots) V = 0$$

when the operators are defined by the equations

$$D = \frac{1}{\sin u}\frac{d}{du}\sin u \frac{d}{du} + \frac{d^2}{dv^2} - \frac{d}{dv}$$

$$D_n = e^{(n+1)v}\left\{\frac{1}{\sin u}\frac{d}{du}\sin nu \frac{d}{du} + \frac{\sin nu}{\sin u}\left(\frac{d^2}{dv^2} + n\frac{d}{dv}\right)\right\}$$

We may then regard the equation (5) as the standard form of Laplace's equation for a surface of revolution given by (1).

It will be convenient to use, also, the co-ordinates ζ, μ defined by

$$(6) \qquad e^v = \zeta, \qquad \cos u = \mu.$$

With this definition $\zeta = 1$ is the surface s and $\zeta = 0$ the sphere at infinity and we therefore require, throughout, solutions which are available for $0 < \zeta \leq 1$ and in particular are evanescent for $\zeta = 0$. At infinity α/ζ and μ reduce to the usual polar co-ordinates r and θ.

We may then write

$$(7) \qquad D \equiv (d/d\mu)(1-\mu^2)(d/d\mu) + \theta_\zeta(\theta_\zeta - 1)$$

where as usual the homogeneous operator is given by

and $$\theta_\zeta \equiv \zeta(d/d\zeta)$$

(8) $$D_n \equiv \zeta^{n+1}\{(d/d\mu)\chi_n(\mu)(1-\mu^2)(d/d\mu) + \chi_n(\mu)\theta_\zeta(\theta_\zeta + n)\}$$

with

(9) $$\chi_n(\mu) \equiv \sin n\,(\cos^{-1}\mu)/\sin(\cos^{-1}\mu).$$

Then $\chi_n(\mu)$ is evidently a series of terms in μ^{n-1}, μ^{n-3}, μ^{n-5} $\cdots$ and in particular

(10) $$\chi_1 = 1,\ \chi_2 = 2\mu,\ \chi_3 = 4\mu^2 - 1.$$

II. *The Standard Form for General Harmonics.*

2. 1. The fundamental case to deal with is when $\alpha_1, \alpha_2 \cdots \alpha_n \cdots$ are all zero. s is then the sphere $r = \alpha$, being in fact the zero v level in the transformation

$$z + i\rho = \alpha e^{iu-v}$$

or the unit ζ level in the transformation

$$z = \alpha\cos u/\zeta, \quad \rho = \alpha \sin u/\zeta.$$

The equation for V becomes simply

$$DV = 0;$$

and since $P_k(\mu)$ satisfies the equation

(11) $$[(d/d\mu)(1-\mu^2)(d/d\mu) + k(k+1)]P_k(\mu) = 0,$$

and ζ^{k+1} the equation

(12) $$[\theta_\zeta(\theta_\zeta - 1) - (k+1)k]\zeta^{k+1} = 0;$$

a harmonic evanescent on S, the sphere at infinity, is therefore available in the form

$$V = P_k(\mu)\zeta^{k+1}$$

recognisable in the usual form

$$V = P_k(\mu), \quad (\alpha/r)^{k+1}$$

since $r = \alpha/\zeta$. We can then write down a solution of Laplace's equation for the sphere which on the spherical surface s is equal to any function of z and ρ, or (as for example in the case of the problem of the freely charged conductor) is constant on s.

2.2. Now it will be convenient to systematize the surface s to be treated by assuming that if we write

$$\alpha_n/\alpha = \gamma_n \lambda^n, \qquad (\lambda^2 < 1)$$

then the coefficients

$$\gamma_1, \gamma_2 \cdots \gamma_n \cdots$$

are all of the same order of magnitude. It will also be possible to deal with the more general case when there exist integers $i(n)$ such that if we write

$$\alpha_n/\alpha = \gamma_n \lambda^{i(n)} \qquad (\lambda^2 < 1)$$

then the γ coefficients are all of the same order of magnitude. Thus we might have

$$(\alpha_1, \alpha_2)/\alpha = (\gamma_1, \gamma_2)\lambda; \quad (\alpha_3, \alpha_4)/\alpha = (\gamma_3, \gamma_4)\lambda^2 \cdots$$

and the method now under discussion could easily be made applicable. However, it is not feasible to treat generally the cases when the functions $i(n)$ are required, though each separate case can readily be treated. It must always be the case that the basic assumption as to dW/dz is still operative, and this will evidently limit the γ coefficients in certain well defined ways.

Suppose now, under the assumption that the γ coefficients are all of the same order of magnitude, we take

$$V = f_0 + \lambda f_1 + \cdots + \lambda^n f_n + \cdots \tag{13}$$

and solve the equations

$$\begin{aligned} Df_0 &= 0 \\ Df_1 &= \gamma_1 D_1 f_0 \\ Df_2 &= \gamma_1 D_1 f_1 + \gamma_2 D_2 f_0 \\ Df_3 &= \gamma_1 D_1 f_2 + \gamma_2 D_2 f_1 + \gamma_3 D_3 f_0 \end{aligned} \tag{14}$$

and in general

$$Df_{m+1} = \gamma_1 D_1 f_m + \gamma_2 D_2 f_{m+1} + \cdots + \gamma_{m+1} D_{m+1} f_0.$$

Then we can develop a harmonic to any degree of approximation in the form

$$v = f_0 + \lambda f_1 \cdots + \lambda^p f_p + O(\lambda^{p+1})$$

and the exact form of any harmonic will be

$$v = f_0 + \lambda f_1 \cdots + \lambda^p f_p + \cdots.$$

To find any harmonic we require therefore to be able to deal with the equations (14).

2. 3. Since D is the usual operator for spherical polar co-ordinates, these equations have the characteristic that the complementary functions are always simply

$$\Sigma a_p P_k(\mu)\zeta^{p+1}.$$

We may also explicitly notice that

$$DP_m(\mu)\zeta^\sigma = (\sigma - m - 1)(\sigma + m)P_m(\mu)\zeta^\sigma \tag{15}$$

in the usual way, for this fact has the important consequence that if in any equation

$$Dv = F(\mu, \zeta)$$

F consists of the sum of such terms as $b_{m,\sigma}P_m(\mu)\zeta^\sigma$ (in which $\sigma - m - 1 \neq 0 \neq \sigma + m$), then a particular integral is available in the form

$$\Sigma c_{m,\sigma}P_m(\mu)\zeta^\sigma$$

where

$$c_{m,\sigma} = b_{m,\sigma}/(\sigma - m - 1)(\sigma + m)$$

and a general solution involving an enumerable infinity of arbitrary constants is given by

$$\Sigma a_p P_p(\mu)\zeta^{p+1} + \Sigma c_{m,\sigma}P_m(\mu)\zeta^\sigma. \tag{16}$$

There is thus no theoretical difficulty whatever in solving any equation when the right side F consists of P functions multiplied by polynomials in ζ.

Now we easily find that

$$D_1 P_m(\mu)\zeta^\sigma = (\sigma - m)(\sigma + m + 1)\zeta^{\sigma+2}P_m(\mu) \tag{17}$$

and we shall shortly show that, in general,

$$D_n P_m(\mu)\zeta^\sigma = \zeta^{\sigma+n+1}[(\sigma, m)^n_{-n+1}P_{m-n+1} + (\sigma, m)^n_{-n+3}P_{m-n+3} + \cdots + (\sigma, m)^n_{n-3}P_{m+n-3} + (\sigma, m)^n_{n-1}P_{m+n-1}]. \tag{18}$$

Then it follows by an inductive argument, that since

$$f_0 = \Sigma g_k \zeta^{k+1}P_k(\mu)$$

the right side in all the equations (14) is of the requisite form for the simple solutions given in (16).

Thus a harmonic in the general case now under discussion takes the standard form

$$v = \Sigma \Pi^m(\zeta) P_p(\mu)$$

where by $\Pi^m(x)$ we mean a polynomial function of x of degree m.

It will be a simple matter in the sequel to investigate the degrees of the polynomials $\Pi(\zeta)$ and to show how in any particular case their actual coefficients can be evaluated.

III. *A Study of the D_n Operators.*

3.1. The operator D_n is given by

$$\begin{aligned} D_n &= \zeta^{n+1}\{(d/d\mu)\chi_n(\mu)(1-\mu^2)(d/d\mu) + \chi_n(\mu)\theta_\zeta(\theta_\zeta + n)\} \\ &= \zeta^{n+1}\{\chi_n(\mu)[(d/d\mu)(1-\mu^2)(d/d\mu) + \theta_\zeta(\theta_\zeta + n)] \\ &\quad + \chi_n'(\mu)(1-\mu^2)(d/d\mu)\} \end{aligned}$$

where

$$\chi_n(\mu) = \sin nu/\sin u = s_{n-1}\mu^{n-1} + s_{n-3}\mu^{n-3} + \cdots.$$

Now since

$$(d/d\mu)(1-\mu^2)(d/d\mu)P_p = -p(p+1)P_p$$

and

$$(2p+1)(1-\mu^2)(d/d\mu)P_p = p(p+1)(P_{p-1} - P_{p+1})$$

therefore

$$\begin{aligned} D_n\zeta^\sigma P_m(\mu) = \zeta^{\sigma+n+1}\{&\chi_n(\mu)[-m(m+1) + \sigma(\sigma+n)]P_m \\ &+ \chi_n'(\mu)\cdot[m(m+1)/(2m+1)](P_{m-1} - P_{m+1})\}. \end{aligned}$$

It is also an important characteristic of P_m functions that

$$(2p+1)\mu P_p(\mu) = pP_{p-1}(\mu) + (p+1)P_{p+1}(\mu),$$

for it then follows that $\mu^2 P_p$ gives terms in P_{p-2}, P_p, P_{p+2} and in general that $\mu^q P_p$ gives terms in P_{p-q}, $P_{p-q+2}\cdots P_{p+q-2}$, P_{p+q}. Evidently therefore $\chi_n(\mu)P_m$ may be expressed in terms of

$$P_{m-(n-1)},\ P_{m-(n-3)}\cdots P_{m+(n-3)},\ P_{m+(n-1)};$$

and, in a similar fashion, $\chi_n'(\mu)P_{m\pm1}$ may be expressed in terms of

$$P_{m\pm1-(n-2)},\ P_{m\pm1-(n-4)}\cdots P_{m\pm1+(n-4)},\ P_{m\pm1+n-2}.$$

Thus in terms of characteristic coefficients $(\sigma, m)_r^n$, where r takes the values

$$-(n-1),\ -(n-3)\cdots(n-3),\ (n-1)$$

$$D_n\zeta^\sigma P_m(\mu) = \zeta^{\sigma+n+1}\sum_r (\sigma, m)_r^n P_{m+r}(\mu).$$

It will prove a severe exercise in Legendre functions to work out the coefficients $(\sigma, m)_r^n$ in general. However any required coefficients can be obtained and we append the value of members of this class of coefficient which will allow a number of applications of the method to be carried out. Thus in addition to the expression for D_1 already given in (17), we give

$$D_2\zeta^\sigma P_m(\mu) = \zeta^{\sigma+3}[(\sigma, m)_{-1}{}^2 P_{m-1} + (\sigma, m)_{+1}{}^2 P_{m+1}]$$

with

$$(2m+1)(\sigma, m)_{-1}{}^2 = 2m(\sigma - m + 1)(\sigma + m + 1)$$
$$(2m+1)(\sigma, m)_{+1}{}^2 = 2(m+1)(\sigma - m)(\sigma + m + 2)$$

and

$$D_3\zeta^\sigma P_m(\mu) = \zeta^{\sigma+4}[(\sigma, m)_{-2}{}^3 P_{m-2} + (\sigma, m)_0{}^3 P_m + (\sigma, m)_2{}^3 P_{m+2}]$$

with

$$(2m-1)(2m+3)(\sigma, m)_0{}^3 = \sigma(\sigma+3)(4m^2+4m-1) - m(m+1)(4m^2+4m-9)$$

and

$$(2m+1)(2m-1)(\sigma, m)_{-2}{}^3 = 4m(m-1)(\sigma - m + 2)(\sigma + m + 1)$$
$$(2m+1)(2m+3)(\sigma, m)_2{}^3 = 4(m+1)(m+2)(\sigma - m)(\sigma + m + 3).$$

IV. *The Construction of General Harmonics.*

4.1. It is now possible to work out harmonics satisfying Laplace's equation in its general form

$$(D - \gamma_1\lambda D_1 - \gamma_2\lambda^2 D_2 \cdots - \gamma_n\lambda^n D_n \cdots)v = 0.$$

The equation for f_0 allows a general solution

$$\Sigma\, a_k P_k(\mu)\zeta^{k+1}.$$

But let us investigate the harmonic—which it will be convenient to call $\Phi_k(\mu, \zeta)$—in which

$$f_0 = P_k(\mu)\zeta^{k+1}.$$

In solving the equations (14) for $f_1, f_2 \cdots f_n \cdots$ which are all of the form

$$Df_n = \Sigma\, b_{m,\sigma} P_m(\mu)\zeta^\sigma,$$

we shall uniformly select the particular integral

$$f_n = \Sigma\, c_{m,\sigma} P_m(\mu)\zeta^\sigma$$

where

$$c_{m,\sigma} = b_{m,\sigma}/(\sigma - m - 1)(\sigma + m)$$

and add no term of the complementary function given in (16). In this way we shall build up a harmonic $\Phi_k(\mu, \zeta)$ in the simplest possible manner. Nevertheless no generality is lost, for by introducing at any stage—say in the solution of the equation for f_n—any term of the complementary function, as for example

$$f_n = \Sigma\, c_{m,\sigma} P_m(\mu)\zeta^\sigma + a_p P_p(\mu)\zeta^{p+1}$$

we shall ultimately arrive at the solution

$$v = \Phi_k(\mu, \zeta) + a_p \lambda^n \Phi_p(\mu, \zeta)$$

and so obtain the sum of multiples of two standard harmonics, instead of one alone.

4. 2. Adopting then the solution

$$_k f_0 = P_k \zeta^{k+1}$$

we have

$$v = \Phi_k(\mu, \zeta) = P_k \zeta^{k+1} + O(\lambda)$$

and since the equation for f_1 is

$$Df_1 = \gamma_1 D_1 f_0 = 2\gamma_1 (k+1) P_k \zeta^{k+3}$$

we therefore take

$$_k f_1 = [(k+1)/(2h+3)]\, \gamma_1 P_k \zeta^{k+3}.$$

Now the equation for f_2 is

$$Df_2 = \gamma_2 D_2 f_0 + \gamma_1 D_1 f_1$$

and as

$$(2k+1) D_2 P_k \zeta^{k+1} = 4\zeta^{k+4}[4k(k+1)P_{k-1} + (k+1)(2h+3)P_{k+1}]$$
$$D_1 P_k \zeta^{k+3} = 6(k+2)\zeta^{k+5} P_k$$

we adopt the particular integral

$$_k f_2 = \zeta^{k+4}[c_k P_k \zeta + c_{k-1} P_{k-1} + c_{k+1} P_{k+1}]$$

with

$$2(2k+3)(2h+5)c_k = 3(k+1)(k+2)\gamma_1^2$$
$$(2k+1)(2h+3)c_{k-1} = 2k(k+1)\gamma_2$$
$$(2k+1)(2h+5)c_{k+1} = (k+1)(2h+3)\gamma_2.$$

In a similar manner, from the equation for f_3

$$Df_3 = \gamma_3 D_3 f_0 + \gamma_2 D_2 f_1 + \gamma_1 D_1 f_2$$

and the facts that

$$D_3 P_k \zeta^{k+1} = \zeta^{k+5}[(k+1,k)_{-2}{}^3P_{k-2} + (k+1,k)_0{}^3P_k + (k+1,k)_2{}^3P_{k+2}]$$
$$D_2 P_k \zeta^{k+3} = 2\zeta^{k+6}[4k(2h+4)P_{k-1} + 3(k+1)(2h+5)P_{k+1}]/(2k+1)$$
$$D_1(P_{k-1}, P_{k+1})\zeta^{k+4} = \zeta^{k+6}[5(2h+4)P_{k-1}, 3(2k+6)P_{k+1}]$$

we deduce the particular integral

$$\begin{aligned} {}_kf_3 = (k+1)\zeta^{k+5}\Big[\Big\{ & \frac{4k^2+6k-1}{(2k-1)(2h+3)(2k+5)}\,\gamma_3 \\ & + \frac{3.5(k+2)(k+3)}{2(2h+3)(2h+5)(2h+7)}\,\gamma_1{}^3\zeta^2 \Big\} P_k \\ & + \gamma_1\gamma_2\zeta\Big\{ \frac{6k(k+2)}{(2k+1)(2h+3)(2h+5)}\,P_{k-1} \\ & + \frac{3}{2(2h+1)(2k+7)}\Big[\frac{(k+1)(2h+5)}{2h+3} + \frac{(k+3)(2h+3)}{2h+5}\Big]P_{k+1} \\ & + \frac{4\gamma_3}{(2k+1)(2k+3)}\Big\{ \frac{k(k-1)}{2k-1}\,P_{k-2} + \frac{(k+2)^2}{2k+7}\,P_{k+2}\Big\}\Big] \end{aligned}$$

and in general

$$\begin{aligned} {}_kf_n = \zeta^{k+n+2}\{ & (\zeta^{n-1}, \zeta^{n-3}\cdots)P_k \\ & + (\zeta^{n-2}, \zeta^{n-4}\cdots)(P_{k-1}, P_{k+1}) + \cdots + (1)(P_{k-n+1}, P_{k+n+1}\}, \end{aligned}$$

where we write

$$(1, \zeta^\sigma \cdots)P_r \equiv (c_{0,r} + c_{\sigma,r}\zeta^\sigma \cdots)P_r$$

denoting by $c_{m,n}$ a specific coefficient. By means of the calculations of the coefficients involved in ${}_kf_1$, ${}_kf_2$ and ${}_kf_3$, harmonics suitable for a number of applications are available in the form

$$\nu = {}_kf_0 + \lambda_k f_1 + \lambda^2{}_kf_2 + \lambda^3{}_kf_3 + O(\lambda^4).$$

It is found that this degree of approximation is adequate for a number of electrical and hydrodynamical problems.

V. *Applications of General Harmonics.*

5.1. It remains now to record the simple manner in which these standard harmonics $\Phi_k(\mu, \zeta)$ can be used to solve the problems mentioned at the beginning of this paper.

All these harmonics are evanescent on the sphere at infinity, since they involve only positive powers of ζ: they are also free from singularities in the region between the surface s and the sphere at infinity. Thus two conditions of the problems are satisfied. The remaining point is to show how harmonics may be obtained which on the surface s, given in our co-ordinates $(\mu\,\zeta)$ as the unit ζ level, take certain specific forms.

Now on the surface s

$$\begin{aligned}\Phi_k(\mu,\zeta) &= \Phi_k(\mu,1)\\ &= P_k+\lambda(P_k)+\lambda^2(P_k,P_{k\pm1})\cdots+\lambda^n(P_k,P_{k\pm1},\cdots,P_{k\pm(n-1)})\end{aligned}$$

where the terms in $\lambda^r P_s$ have specific coefficients. Thus to find a harmonic which reduces to a multiple of P_k on the surface s we take

$$\begin{aligned}v = {}& \Phi_r+\lambda^2(\Phi_{k\pm1})+\lambda^3(\Phi_{k\pm1},\Phi_{k\pm2})\\ &+\cdots+\lambda^n(\Phi_{k\pm1},\Phi_{k\pm2},\cdots,\Phi_{k\pm(n-1)})\end{aligned}$$

(where again each term has a specific coefficient) choosing the coefficients in such a way as successively to make the P_{k+1}, P_{k-1}, P_{k+2}, $P_{k-2}\cdots$ terms in V disappear, when we successively take

$$\begin{aligned}V &= \Phi_k+O(\lambda^2)\\ V &= \Phi_k+\lambda^2(\Phi_{k\pm1})+O(\lambda^3)\\ V &= \Phi_k+\lambda^2(\Phi_{k\pm1})+\lambda^3(\Phi_{k\pm1},\Phi_{k\pm2})\end{aligned}$$

and so on.

This procedure allows harmonics to be constructed which on the surface s take any specific form of the type

$$C_0P_0(\mu)+C_1P_1(\mu)+C_2P_2(\mu)\cdots+C_kP_k(\mu)+\cdots.$$

Conclusion.

We do not now proceed any further with the study of these standard harmonics for the general surface of revolution under consideration. We have fully investigated their structure. The interesting fact that they form series of P_k functions multiplied by polynomials makes it clear that no theoretical reason exists for abandoning the search for harmonics however general the surface in relation to which they are to be used, provided only that the equation of the surface can be thrown into the general form here treated. The only difficulty in solving the classical problems of electrostatics and

hydrodynamics which we have cited, to any degree of approximation whatever, resides in the heavy arithmetic which may well prove irksome. However, if the applications warrant the trouble involved, there is no reason why the coefficients of these harmonics should not be tabulated, much as the values of functions which are useful in physics have been tabulated. This work could be carried out by arithmeticians with no knowledge of mathematics.

The important point however from our point of view is that these harmonics exist and have a simple structure. This simplicity largely depends upon the fact that each type of surface leads on to harmonics available for rather more general surfaces which are, nevertheless, still merely sets of $P_k(\cos u)$ functions multiplied by powers of e^v. The general determination of the coefficients involved is possibly hardly feasible, though investigations in connexion with them easily yield interesting theorems about the functions.

A Continuum Every Subcontinuum of which Separates the Plane.

By G. T. WHYBURN.*

1. *Introduction.* In this paper there will be constructed an example of a continuum of the type, hitherto unknown, indicated in the above title. Indeed, we shall construct a compact plane continuum M having the following properties:

1). *M is the common boundary of two domains;*

2). *every subcontinuum of M separates the plane;*

3). *every subcontinuum of M contains a continuum which is homeomorphic with M, or, in other words, M is topologically contained in each of its subcontinua;*

4). *M contains no uncountable collection of mutually exclusive subcontinua and therefore no indecomposable continuum; obviously it contains no arc;*

5). *M admits of upper semi-continuous decomposition† into elements (continua or points) all save a countable number of which are points and with respect to which M is a simple closed curve; clearly, then, all the "point elements" in this decomposition are local separating points of M and are points of ordinary order 2 of M;*

6). *M contains two continua which are not homeomorphic with each other.*

In addition, it will be shown, in general, that any compact plane continuum having property 2) contains a continuum having properties 1) and 2) and that every continuum having 1) and 2) also has properties 4), 5), and 6). From these facts it follows that no plane continuum which separates the

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† That is, an upper semi-continuous collection G of subsets g of M exists such that every point of M belongs to exactly one element of G. A collection G of point sets is upper semi-continuous provided that if g is any element of G and $g_1, g_2, g_3, \cdots$ is any infinite sequence of elements of G containing points $P_1, P_2, P_3, \cdots$; respectively, such that the sequence of points $P_1, P_2, P_3, \cdots$ has a sequential limit point belonging to g, then g contains the entire sequential limiting set of the sequence of sets $g_1, g_2, g_3, \cdots$. See R. L. Moore, "Concerning Upper Semi-continuous Collections of Continua," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 416-428.

plane can be homeomorphic with every one of its subcontinua. So far as the author knows, no example, outside of that of a simple continuous arc, is known of a continuum which is homeomorphic with every one of its subcontinua,* and prior to the example given in this paper, no continuum, other than an arc, is known which is topologically contained in every one of its subcontinua.

To give a general idea of the construction of the continuum M, we list still another property of M, which is stronger than property 5), as follows:

7). *M can be decomposed into a contracting † collection G of mutually exclusive elements every one of which is either a point or a continuum of condensation of M which is homeomorphic with M and such that with respect to the elements of G, M is a simple closed curve, and the elements of G which are not points are dense in M.*

We begin, then, with the idea in mind to construct a continuum M having property 7) and also such that a) every element of G which is not a point is a continuum of condensation of every subcontinuum of M of which it is a proper part, and b) if $E_1, E_2, E_3 \cdots$ is a sequence of continua such that E_1 is an element of G and, for $n > 1$, E_n is an element of the collection of elements in E_{n-1} corresponding to the collection G in M, then the diameter of E_n approaches zero as n increases indefinitely. It will be apparent from the construction and proof given below that our continuum M has these properties. And it is true, furthermore, that any compact plane continuum having properties 7), a), and b) must also have properties 1)—6).

Very generally speaking, M is constructed as follows. We take a square Q, which we call the germ cell for M. Take a non-dense perfect set R on Q whose complementary segments are all of length $\leqq 1/9$ the side of Q. Within Q and in the proximity of each complementary segment s of R we introduce a germ cell square $q(s)$ of side $1/9$ the length of s, which will be developed exactly like the square Q, and then subject s to a certain *prolongation* process, in such a way that the prolongation of s approaches asymptotically the entire cell which is developed from $q(s)$. The prolongation process is, of course, carried out simultaneously with the development of $q(s)$. At the same time as s is being deformed or prolonged by stages,

* In this connection, see J. R. Kline, "Separation Theorems and Their Relation to Recent Developments in Analysis Situs," *Bulletin of the American Mathematical Society*, Vol. 34 (1928), p. 192.

† A collection of point sets is contracting if, for each $\epsilon > 0$, at most a finite number of elements of the collection are of diameter $> \epsilon$. This terminology is due to R. L. Moore.

the previous prolongations of s are divided into intervals and these developed in the same way as is a side of Q. After this process is carried to completion, the continuum M is defined as the sum of the residue R of Q, the corresponding residues on the deformations of the deformed segments in Q, together with the limit points of these deformations. As so constructed and defined, M will admit of a decomposition of the type in property 7), the elements of G being the cells such as $q(s)$ and corresponding cells resulting from the deformation of segments from the prolongation of s, and from the prolongations of these segments, and so on, together with all points in the residues above mentioned and all of the added limit points not in some cell element.

2. *Development Stages.* We first define some terms which it will be convenient to use for the sake of brevity. A *coupling joint* of bore b and length h is a set congruent to the sum of the two intervals from $(0,0)$ to $(h,0)$ and from $(0,b)$, to (h,b). An ordinary *elbow joint* of bore b and branch lengths h and h' is a set congruent to the sum of the two broken lines joining, in order, $(0,h)$, $(0,0)$, $(h',0)$ and (b,h), (b,b), (h',b) respectively. A *reducing elbow joint* of ratio $b:b'$ is a set congruent to the sum of the two broken lines joining, in order, the points $(0,h)$, $(0,0)$, $(h',0)$ and (b,h), (b,b'), (h',b') respectively. In the construction below, the bores and lengths of the joints used are usually predetermined by the position into which the joints must fit, and hence they are given below only when this is not the case. All reducing elbows used are of ratio 9 : 1. A coupling joint is said to be *rectangular* with a segment s when either of its sides may be joined by lines to s so as to form a rectangle.

A *loop* of *type 0* in a segment s of length L is simply a coupling joint of bore L and length L, which is rectangular with s and on the side of s opposite to the deformations of s at a distance L from s. A loop of type 1 in such a segment s is a coupling joint of bore L/9 and length L which is rectangular with s and is on the side of s opposite to the deformations of s and at a distance $L/9$ from s.

A segment s is said to be in *stage 0* of deformation when it is chosen for deformation. A segment s is put into stage 1 of deformation as follows. Choose co-ordinates with the origin at the left end point of s, X-axis along s and unit of length so that the length L of s is 9. As the central germ cell square for s take the square with vertices $(4,4)$, $(4,5)$, $(5,5)$, and $(5,4)$. Then for the first prolongation of s, take the sum of the two broken lines joining, in order, the points $(0,0)$, $(0,3)$, $(2,3)$, $(2,7)$, $(7,7)$, $(7,4)$, and

(9, 0), (9, 3), (3, 3), (3, 6), (6, 6), (6, 4), respectively. The width or *bore* of the first prolongation of s is $L/9$.

An interval I is put into *stage 1* of *interval development* by dividing it into 9 equal subintervals, these ordered from one end to the other, and the four alternate segments thus formed, beginning with the second, chosen for deformation.

A square (or germ cell) is put into *stage 1* of *cell development* by putting each of its sides into stage 1 of interval development, thus forming a *growing cell in stage 1.*

Now assuming that loop types and deformation and development stages have been defined for all integers $< n$, $(n > 1)$, we proceed to define these notions for n as follows.

A segment s of length L which is in stage $n-1$ of deformation is put into stage n as follows:

a). its central growing cell is put into stage $n-1$;

b). its $(n-1)$st prolongation is divided into a set G_{n-1} of intervals all of length $L/9^{n-1}$, i. e., equal to the bore of this prolongation;

c). for each integer $k \leqq n-1$, the intervals of the set G_k are all put into stage $n-k$ of development, with all deformations of segments placed on the outside of the kth prolongation of s, i. e., on the side opposite to that occupied by the other side of this prolongation;

d). in its germ cell square, in each segment which has been selected for deformation and whose deformation is in the jth stage, $j \leqq n-1$, introduce a loop of type j. Order these loops clockwise, beginning with the lower right vertex of the cell, into a sequence $q_1, q_2, \cdots, q_{4r}$ $(r = 5^{n-1}-1)$. Now it will be seen from the general definition of loops given below that the bore of each of these loops is $L/9^n$. The bore of the $(n-1)$st prolongation of s being $L/9^{n-1}$, we may join this prolongation of s to q_1 by a reducing elbow with reducing ratio 9:1. For each $i = mr$, $(m = 1, 2, 3)$, join the more advanced end of q_i to the less advanced end of q_{i+1} by an ordinary elbow joint. For every remaining $i < 4r$, join the more advanced end of q_i to the less advanced end of q_{i+1} by a coupling joint. Finally, attach a coupling joint of length $L/9^n$ on to q_{4r}. Add together the $4r$ loops $[q_i]$ and the $4r+1$ joints, and call the set thus obtained the nth prolongation of s and denote it by $p_n(s)$. Clearly $p_n(s)$ is the sum of two broken lines which will be called its two sides, the one nearer to the central cell being called the *near side* and the other the *far side* of $p_n(s)$. The bore of $p_n(s)$ is $L/9^n$.

For $n > 1$, a loop of type n is introduced into a segment s of length L

which is in the nth stage of deformation according to the following procedure. In each of the segments chosen for deformation on the sum of the first $n-1$ prolongations of s whose deformation is in the rth stage, $r \leqq n-1$, introduce a loop of type r. Let these loops be ordered, beginning with the initial point of the near side of $p_1(s)$, which is the right end point of s, and proceeding in order along the near sides of $p_1(s)$, $p_2(s) \cdots$ to the final end of $p_{n-1}(s)$, in a sequence $q_1, q_2, \cdots, q_k$. Then beginning with the final end of the far side of $p_{n-1}(s)$, proceed backwards, in order, along the far sides of $p_{n-1}(s)$, $p_{n-2}(s) \cdots$ to the initial end of the far side of $p_1(s)$, (the right end point of s), with the sequence $q_{k+1}, q_{k+2}, \cdots, q_m$. Now the bore of each of these loops $[q_i]$ is $L/9^n$. For we have assumed that for all integers $r < n$ and each segment s' of length h which is in the rth stage of deformation, a loop of type r in s' is of bore $h/9^r$. And since for each segment s' corresponding to a loop q_i and being in stage r, $h = L/9^{n-r}$, it follows that the bore of the loop q_i corresponding is $1/9^r \cdot L/9^{n-r} = L/9^n$. Now for each i, $1 < i < m$ and $i \neq k$, join q_i to q_{i+1} by a coupling joint or an ordinary elbow according as the segments corresponding are or are not co-linear. Join q_k to q_{k+1} by a joint which is the sum of two congruent elbow joints of branch lengths $h = L/9^n$, $h' = 7L/2 \cdot 9^n$. On to q_1 and q_m attach elbow joints of branch lengths $h = 3L/9^n$, $h' = 2L/9^n$, the branch of length h being attached to q_1 and q_m respectively and the joints being turned so that the right end point of s is in the angle of the one attached to q_1 and the left end point of s in the one attached to q_m. Add together these m loops $[q_i]$ and $m+1$ coupling joints, and the set thus obtained is a loop of type n introduced into s. Its bore, as shown above, is $L/9^n$.

For $n > 1$, an interval I in stage $n-1$ of interval development is put into stage n by putting each of the four segments selected for deformation in stage 1 into stage n of deformation, and by placing each of the five complementary intervals of these segments into stage $n-1$ of interval development.

For $n > 1$, a growing cell C in stage $n-1$ of cell development is put into stage n of cell development by putting each of the four sides of its germ cell square into stage n of interval development.

3. *Construction and definition of the continuum M. Proof that M has properties* 1), 5) *and* 7). To construct the continuum M we take as a germ cell square any square Q in the plane. Then subject Q successively to stages 1, 2, 3, $\cdots$ of cell development as defined in § 2.

In order to define M we first define certain functions, or operators, p_n, p_n', p, p', S, g_n, g_n', d, and d'. For each segment s which at some stage in the

development of M is selected for deformation, let $p_n(s)$ denote the nth prolongation of s; let $p_n'(s)$ denote the residue of $p_n(s)$, i. e. $p_n(s)$ minus all segments which are selected at some stage from $p_n(s)$ for deformation; let $p(s) = \sum_1^\infty p_n(s)$, the total prolongation of s, and $p'(s) = \sum_1^\infty p_n'(s)$, the residue of $p(s)$. Now for any set X, which may be either a square, a pro-

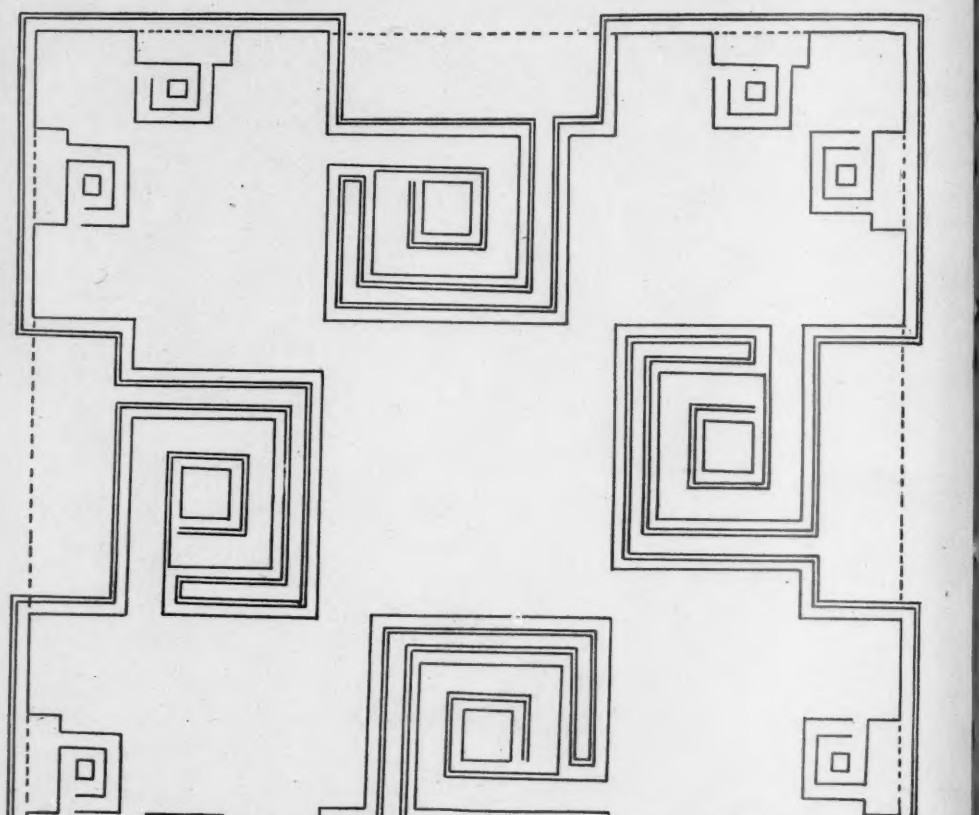

FIG. 1. THE FOURTH PROLONGATION OF A SEGMENT s, WITH CENTRAL GROWING CELL IN STAGE 3 OF DEVELOPMENT. (This figure is not drawn to scale and does not correspond exactly to the description given for the development of M; the sides of the germ cell square for s are divided into 3 parts rather than 9 as in the description.) The segment s is not shown; segments of the type marked s' are in stage 1 of deformation and have corresponding loops of type 1; segments such as the one marked s'' are in stage 2 of deformation and have corresponding loops of type 2.

longation p of some segment, or a collection g of such prolongations, let S denote the operation of selecting a segment from the set X which, at some stage in the development of M, has been chosen for deformation; and denote the segment, which is the result of the operation S or is the value of S, by $S(X)$. For each segment s which, at some stage in the development of M, is chosen for deformation, let $g_1(s)=p(s)$, $g_1'(s)=p'(s)$, and for every $n>1$, let $g_n(s)=\sum_{S \text{ in } g_{n-1}} p[S(g_{n-1})]$, and $g_n'(s)=\sum_{S \text{ in } g_{n-1}} p'[S(g_{n-1})]$, the residue of $g_n(s)$. Finally, let $d(s)=\sum_1^\infty g_n(s)$; $d'(s)=\sum_1^\infty g_n'(s)$, the residue of $d(s)$; let R denote the residue of Q, i. e., Q minus all segments selected from Q for deformation, let $F=R+\sum_{s \text{ in } Q} d'(s)$, and let $\bar{F}=M$, the completed cell.

For every square q selected in the development of M as the germ cell of some segment s, it is clear that the completed cell, which we shall denote by $C(s)$, defined with respect to q just as $\bar{F}$ is defined relative to Q, is homeomorphic with M; indeed it is identical in structure with M except in size, and is a subset of M and indeed of $\bar{F}-F$, except when $q=Q$. Let C denote the collection of all completed cells $C(s)$, such that for some segment s selected in Q for deformation, for some n $(1 \leq n < \infty)$, and for some S in $g_n(s)$, $C(s)=C\{S[g_n(s)]\}$, and let T denote the sum of all the point sets of the collection C, i. e.,

$$T=\sum_{s \text{ in } Q} \sum_{n=1}^{\infty} \sum_{S \text{ in } g_n(s)} C\{S[g_n(s)]\}.$$

Now for every element c of C, there exists an integer n and a sequence $s_1, s_2, s_3, \cdots, s_n$ of segments selected for deformation in the development of M, such that (α) s_1 belongs to Q, (β) for every i, $1<i\leqq n$, s_n is in $p(s_{n-1})$, and (γ) $c=C(s_n)$. Conversely, for every such sequence of segments satisfying (α) and (β), it is clear that $C(s_n)$ is an element of C. Thus C is identically the collection of all cells c completed in the construction of M such that a sequence $s_1, s_2, \cdots, s_n$ of segments selected for deformation exists satisfying conditions (α), (β), and (γ).

Now R is a null dimensional perfect set. And for each segment s selected for deformation, and for each n, $p_n'(s)$ is a null dimensional perfect set. Thus $p'(s)$ is an F_σ and indeed $p'(s)+C(s)$ is closed. Hence it follows that for every n, $\sum_{i=1}^{n} g_i'$ is a null dimensional F_σ and that $\sum_1^n g_i' + \sum_{s \text{ in } \Sigma_1^n g_i} C(s)$ is closed. This follows easily by induction from $g_i'(s)=p'(s)$ and the fact that the construction is carried out so that the collection of all prolongations

$[p(s)]$ is a contracting collection, i. e., at most a finite number of its elements are of diameter $>$ any preassigned positive number. Thus for every such s, $d'(s)$ is a null dimensional F_σ and therefore so is $F = R + \sum_{s \text{ in } Q} d'(s)$.

I shall next show that M is a continuum. By definition it is closed and compact. And from the definition of F it is obvious that if A is the center of Q and B a point without Q, then M is the boundary of both R_a and R_b, the components of the complement of M containing A and B respectively. It is sufficient, then, to show that R_a and R_b are different. If not, then there exists an arc AB from A to B with $AB \cdot M = 0$. Let X_1 be the first point on AB, in the order A, B, of $AB \cdot Q$. Since X_1 does not belong to M and thus not to R, it belongs to a segment s_1 in Q which was selected for deformation in the development of M. Now $Q - s_1 + p(s_1) + C(s_1)$ is a continuum K_1 separating A and X_1. Let X_2 be the first point on AB, in the order A, B, which belongs to this continuum. Then as X_2 is within Q and $C(s_1) \subset M$, X_2 belongs to $p(s_1)$. But since it does not belong to $p'(s_1)$, it belongs to some segments s_2 selected for deformation in the development of M. Now $K_1 - s_2 + p(s_2) + C(s_2)$ is a continuum K_2 separating X_2 and A; let X_3 be the first point of $AB \cdot K_2$ in the order A, B, and so on. Continuing this process indefinitely, we obtain a sequence of mutually exclusive segments $s_1, s_2, \cdots$ which were chosen for deformation in the development of M, and such that for every i, $s_i \cdot AB \neq 0$. But for each i, the end points A_i and B_i of s_i belong to F and hence to M; and since $\lim_{i \to \infty} \delta(s_i) = 0$, therefore every limit point of $\sum X_i$ must belong to M. But this set has at least one limit point P; and since $\sum X_i \subset AB$, it follows that $AB \cdot M \supset P$, contrary to supposition. Therefore R_a and R_b are different, and hence M is a continuum which is the common boundary of the two domains R_a and R_b.

Now by what has been proved above, we may divide M into three sets T, F, and X such that $T + B + X = M$, $T \cdot F = T \cdot X = F \cdot X = 0$, $T + X = \bar{F} - F$. Let G denote the collection whose elements are the elements of C together with all points of $F + X$. Then the elements of G are mutually exclusive, the sum of all of them is M, G is a contracting collection (since C is such a collection), and hence also is upper semi-continuous. I shall now show that every element P of G is a continuum or point of order * two

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* A point P of a continuum M is a point of order two of M, in the Menger-Urysohn sense, if for each $\epsilon > 0$, P can be ϵ-separated in M by some two points of M. A point, or continuum, P in M is said to be of order two in M relative to some collection Q of subsets of M if for each $\epsilon > 0$, P can be ϵ-separated in M by two of the elements of Q i. e., two elements A and B of Q exist such that $M - (A + B)$ is the sum of two mutually separated sets M_1 and M_2, where $M_1 \supset P$ and $\delta(M_1 + A + B) < \delta(P) + \epsilon$, [$\delta(x)$ denotes the diameter of the set x].

of M relative both to F and C. If P is an element of C, then there exists a segment s chosen for deformation in the development of M and such that $P = C(s)$. And for any $\epsilon > 0$, an integer k exists so that every point of $\sum_{n=k}^{\infty} p_n(s)$ is at a distance $< \epsilon/2$ from $P = C(s)$. Thus it is clear that if we choose one point of F on each side of $p_k(s)$, or if we choose one segment on each side of $p_k(s)$ which is deformed, then the two points of F so chosen will ϵ-separate P in M and so also will the sum of the two cells corresponding to the two segments thus chosen. If P is a point of F, then either it belongs to R or these exists a segment s which has been chosen for deformation and such that P belongs to $p'(s)$. In either case, for any $\epsilon > 0$, if P is not an end point of any segment chosen for deformation, we can find two points of F, one on either side and belonging to R or to $p'(s)$, respectively, and which ϵ-separate P in M; if P is an end point of some segment s', we can take one of the points on $p_1'(s')$ and the other in R or $p'(s)$ and ϵ-separate P in M, because P is an end point of at most one such segment. In an entirely similar way we can, for each $\epsilon > 0$, ϵ-separate P in M by two elements of C. Finally, if P is a point of X, then since, as is easily seen, $R + \sum_{s \text{ in } Q} \overline{d'(s)} = M$, there exists a segment s_1 in Q such that $P \subset \overline{d'(s_1)}$. Again since, as is easily seen, $\overline{d'(s_1)} = p'(s_1) + C(s_1) + \sum_{s \text{ in } p(s_1)} \overline{d'(s)}$, and P belongs neither to $p'(s_1) \subset F$ nor to $C(s_1) \subset T$, there exists a segment s_2 in $p(s_1)$ such that $P \subset \overline{d'(s_2)}$. Likewise $\overline{d'(s_2)} = p'(s_2) + C(s_2) + \sum_{s \text{ in } p(s_1)} \overline{d'(s)}$, and so on. Continuing this process indefinitely, we obtain a sequence of segments $s_1, s_2, s_3, \cdots$ such that $P \subset \prod_{1}^{\infty} \overline{d'(s_i)}$. But from the method of development of M, the diameter of $\overline{d'(s_i)}$ approaches zero as i increases indefinitely. Thus $P = \prod_{1}^{\infty} \overline{d'(s_i)}$; and since for each i, $\overline{d'(s_i)} \cdot \overline{M - \overline{d'(s_i)}} = A_i + B_i$, where A_i and B_i are the end points of s_i, it follows that P is a point of order two of M relative to F. Now if $\epsilon > 0$, there exists an i such that $A_i + B_i$ ϵ-separates P in M; and if we choose two segments e and f, one on each side of $p(s_i)$ such that their sum separates s_i and s_{i+1} in $p(s_i) + s_i$ and such that these segments were chosen for deformation in the development of M, then it is clear that $C(e)$ and $C(f)$ are elements of C and that their sum ϵ-separates P in M.

It has just been shown that every element of G is a continuum or point of order two of M both relative to F and to C. Since M is a compact continuum of elements of the upper semi-continuous collection G of continua and every element of G is an element of order two, it follows that M is a

simple closed curve with respect to the elements of G. Thus since every element of G is either a point or a cell of C which clearly is homeomorphic with M and is a continuum of condensation of M, it follows that M has property 7). Property 1) has already been demonstrated for M, and property 5) is weaker than 7). Therefore M has properties 1), 5), and 7).

4. *Proof of properties* 3), 2), *and* 6) *for* M. We shall first show that M has property 3). To this end we establish the following:

LEMMA. *If a subcontinuum* N *of* M *contains a point of some cell* E *of* C *and also a point not in* E, *then* N *contains* E.

By the definition of the collection C, it follows that there exists a segment s, chosen for deformation in the development of M, such that $E = C(s)$. Now since N contains a point both of $C(s)$ and of $M - C(s)$, it is readily seen that there exists an integer n such that, for every $m > n$, either the near side or the far side, say the near side, of $p_m'(s)$ is a subset of N. But every point of $C(s) = E$ is a limit point of the sum of the residues of the near sides of $p_m(s)$ for $m > n$. Then since N is closed, it must contain E; and the lemma is established.

Now to prove property 3), let N be any subcontinuum of M whatever. Since, as shown in § 3, every point of $F + X$ is a point of order two of M relative to C, it follows that N contains a point in at least one element E_1 of C. And by the lemma, either $N \supset E_1$ or $N \subset E_1$. If the former of these is true, then property 3) is true. If the latter is true, then since E_1 is homeomorphic with M, it follows by the same argument and lemma as above that an element E_2 of the collection in E_1 corresponding to the collection C in M exists such that either $N \supset E_2$ or $N \subset E_2$. If the former, then property 3) is true, because E_2 is homeomorphic with M. If the latter, then an element E_3 in the collection E_2 corresponding to C in M exists such that either $N \supset E_3$ or $N \subset E_3$; and so on. Continue this process. If N contains any of the continua $E_1, E_2, E_3, \cdots$, then since each of these is homeomorphic with M, property 3) hold for M. If not, then $N \subset \prod_1^\infty E_i$. But clearly this is impossible, because $E_i \supset E_{i+1}$, for every i, and $\delta(E_{i+1}) < (1/9)\delta(E_i)$, and therefore $\prod_1^\infty E_i$ contains only one point. Therefore, for some i, $N \supset E_i$, and property 3) is established.

Since M is non-dense and separates the plane, and since every subcontinuum N of M contains a continuum H which is homeomorphic with M, it

follows that H separates the plane and, as N is non-dense, so also does N. Thus M has property 2).

That M has property 6) follows rather obviously from the fact already established that it has property 7). For if t is any simple continuous arc of elements of G and P is an ordinary point which is an interior element of t, then P is a cut point of the continuum K which is the sum of all the elements of G belonging to t. But M has no cut points, and therefore M and K are not homeomorphic with each other. Indeed it is interesting to note that M *contains exactly* 4 *distinct types of continua.* These are (1) continua homeomorphic with M, (2) continua homeomorphic with arcs of elements of G both of whose end elements are ordinary points, (3) continua homeomorphic with such arcs both of whose end elements are cells of C, and (4) continua homeomorphic with such arcs one end element of which is a point and the other a cell of C. That these are all possible types follows at once with the aid of the lemma in this section. It should be remarked, however, that although every continuum having property 2) contains at least four, one can construct continua having properties 1)—7) and which contain more than four types of continua—indeed containing infinitely many.

Thus we have shown that M has properties 1), 2), 3), 5), 6), and 7). That M also has property 4), although immediately deducible from the construction of M, follows also from the results in the next section concerning continua in general which have properties 1) and 2).

5. *General theorems on continua having property* 2). Let M be any plane continuum having property 2), and let G be any collection of mutually exclusive subcontinua of M. Since each element of G separates the plane, it follows by a theorem of Kuratowski's * that if G is uncountable, it contains a continuum K no proper subset of which separates the plane; but since K contains a proper subcontinuum N, this is contrary to the fact that every subcontinuum of M separates the plane. Hence G must be countable, and thus we have the following theorem:

THEOREM 1. *If every subcontinuum of a plane continuum M separates the plane, then M contains no uncountable collection of mutually exclusive continua. Therefore every subcontinuum of M is decomposable.*

If M is any plane continuum having property 2), it is obvious that every subcontinuum of M also has property 2). And since M separates the plane, clearly it contains a bounded continuum K which is the common

* "Sur les coupures irréductible du plan," *Fundamenta Mathematicae*, Vol. 6 (1924), pp. 130-146, Theorem 8.

boundary of two domains and which, therefore, has properties 1) and 2). Now since, by Theorem 1, K contains no indecomposable continuum, and since no continuum whatever is the sum of a finite or countable number of continua of condensation of itself, it follows that K is not the sum of a finite or countable number of indecomposable continua and continua of condensation of K. Therefore, by results due to Kuratowski,* K may be decomposed upper semi-continuously into a collection G of mutually exclusive elements (continua or points) with respect to which K is a simple closed curve. By Theorem 1, all save a countable number of the elements of G must be points. And clearly each element of G which is a point is local separating point of M and is a point of ordinary order 2 of M. Thus we have the following theorem:

THEOREM 2. *If K is a plane bounded continuum which is the common boundary of two domains and every subcontinuum of K separates the plane, then K admits of upper semi-continuous decomposition into a collection G of elements, (continua or points) all save a countable number of which are points, and with respect to which M is a simple closed curve. All the "point elements" of G are local separating points of K and are points of order 2 of K.*

Now if t is any arc of elements of G, and T is the set of all points of K each of which belongs to some element of G in t, then since, as is evident from Theorem 2, the continuum T has a cut point whereas K does not, then K and T cannot be homeomorphic. Thus every continuum M having property 2) also has property 6). Indeed, we may state the following theorem:

THEOREM 3. *Every plane continuum which separates the plane contains two continua which are not homeomorphic with each other.*

For if this is not so, then M must be homeomorphic with each of its subcontinua. And as M separates the plane, then † every subcontinuum of M must separate the plane, and thus M must have property 2). But then, as shown above, M contains two continua K and T which are not homeomorphic with M, contrary to supposition.

VIENNA, DECEMBER, 1929.

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* "Sur la structure des frontières communes à deux régions," *Fundamenta Mathematicae*, Vol. 12 (1928), pp. 20-42.

† See Brouwer, *Mathematische Annalen*, Vol. 72 (1912), p. 422.

On Continuous Curves and the Jordan Curve Theorem.†

By Leo Zippin.

We shall show that a continuous curve ‡ which satisfies the Jordan Curve Theorem § non-vacuously ¶ is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected (possibly vacuous) point set. Such surfaces we have previously introduced (J. M. T.) and called cylinder-trees: they represent a generalisation of the simple closed surface and of the number-plane. We shall consider further characterisations of them, equivalent in the presence of the continuous curve spaces to the postulation of the Jordan Curve Theorem, and achieve in particular a considerable reduction of R. L. Moore's axiom-system, $\Sigma 1$, for the number-plane.‖

1. We shall have need of two general theorems.

Theorem 1. *If a continuum G^* not a continuous curve is a subset of a continuous curve J^*, then G^* has a subcontinuum G which is not a continuous curve such that G is a subset of a maximal cyclic curve J of J^*.*††

We shall content ourselves with an indication of the proof.‡‡ By the

† Presented to the American Mathematical Society, August 30, 1929.

‡ A metric, locally compact, connected, and connected im kleinen space: for definitions of these terms the reader is referred to Hausdorff's "*Mengenlehre*," 1927.

§ If K is a simple closed curve of a space S, then $S - K$ is the sum of two connected sets without common point such that K is the boundary of each of them.

¶ That is, contains at least one simple closed curve. Compare § 9 of my paper (J. M. T.): "A Study of Continuous Curves and their Relation to the Janiszewski-Mullikin Theorem," *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 744-770.

‖ (F. A.): "On the Foundations of Plane Analysis Situs," *Transactions of the American Mathematical Society*, Vol. 17 (1916), pp. 131-164.

†† For definitions and theorems relating to cyclicly connected continuous curves the reader is referred to the interesting papers of G. T. Whyburn: "Cyclicly Connected Continuous Curves," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38 (C. C.) and "Concerning the Structure of a Continuous Curve," *American Journal of Mathematics*, Vol. 50 (1928), pp. 167-194 (C. S.). The theorems of these papers which we shall use are valid in continuous curve spaces, as we have defined them above.

‡‡ Although the author is not acquainted with any previous statement of the above theorem, it seems to him to be implied in any proof of the sufficiency condition of Theorem 28 (C. S.) which is stated without proof.

theorem that a continuum M_n of a continuous curve J^* is a subset of a continuous curve M^*_n of J^* whose upper distance from M_n may be supposed less than any preassigned e, it is possible to replace the continua (M_n) of G^* (of the Moore-Wilder lemma) by independent arcs (L_n) of J^* with the same sequential continuum of condensation M_a, each arc L_n having at least two points in common with the corresponding M_n. By methods of (C. C.) it is possible to show that infinitely many of these arcs have subarcs in common with a maximal cyclic curve J of J^*, and that infinitely many of the continua of (M_n) as well as M_a have a subcontinuum of points on J. The product of a maximal cyclic curve J of J^* and a continuum G^* of J^*, if it consists of more than a single point, is a continuum G. Since G contains subcontinua of infinitely many of the continua of (M_n) as well as of M_a, it is readily shown that G cannot be connected im kleinen.

THEOREM 2. *If B is a compact subset of a continuous curve S such that the maximal connected subsets of B are points or arcs, and no inner point of any arc b of B is a limit point of $B - b$, then there exists an acyclic continuous curve V of S which contains B.*

Let B^* be the set of points of B which are either components of B or are endpoints of arcs of B. Then B^* is closed and totally disconnected since no inner point of an arc of B can be a limit point of B^*. There is an acyclic continuous curve V^* of S which contains B^*.† Then $T = V^* + B$ is closed and connected, since V^* has at least one point in common with each component of B. Suppose that any subcontinuum G of T fails to be connected im kleinen: then G contains a continuum M of points at which it fails to be connected im kleinen. Suppose that any point m of M fails to be a point of B. Then in some neighborhood of m, G is locally identical with a subcontinuum of V^* and necessarily connected im kleinen at m, since V^* is an acyclic continuous curve. If, on the other hand, every point of M is a point of B, then M being a continuum of points of B is a subarc of B. Let m' be an inner point of the arc M. There is a neighborhood $U_{m'}$ of m' such that $B \cdot U_{m'} \subset M$. If m' fails to be a point of V^*, then in some neighborhood of m', G is locally identical with M and therefore connected im kleinen at m'. If m' is a point of V^*, let $U^*_{m'}$ be a neighborhood of m' such that any point of $M \cdot U^*_{m'}$ can be joined to m' by a subarc of $M \cdot U_{m'}$. Let $U''_{m'}$ be a neighborhood of m' such that any point of $V^* \cdot U''_{m'}$ can be joined to m' by an arc of $V^* \cdot U^*_{m'}$. It is readily seen that any point of

† (J. M. T.) Theorem 1.

$G \cdot U''_{m'}$ can be joined to m' by an arc of $G \cdot U_{m'}$. Then G is a continuous curve. Let V be a continuum of T irreducible about B. Then V is a continuous curve. If V contains any simple closed curve K, a point q of K is not a limit point of B and there is an arc Q of K no point of which belongs to B. Since the arc Q contains at least one non-cutpoint of V, it readily follows that V is not irreducibly connected about B. Then V is an acyclic continuous curve.

2. We assume that a continuous curve S has the following properties:

(A) *S contains at least one simple closed curve.*

(B) *if L is an arc of a simple closed curve of S, then $S - L$ is connected.*

(C) *if K is a simple closed curve of S, then $S - K$ is not connected.*

Since S contains at least one simple closed curve it contains a maximal cyclic curve J, and if S is not J then there exists on J a point x which is a cutpoint of S. Then x belongs to a simple closed curve of J and this contains an arc L containing x such that $S - L$ is not connected. Therefore S must have the further property:

(D) S is cyclicly connected. We have shown previously † that (B, C, D) imply that a continuous curve has the property:

(E) *If K is a simple closed curve of S, $S - K$ is the sum of two maximally connected sets (K-domains) such that K is the complete boundary of each of these.* And, further, the property:

(F) *If K is a simple closed curve of S and M is either of its K-domains (domains relative to K), and $\langle amb \rangle$ ‡ is an open arc of M whose endpoints are on K, then $M - \langle amb \rangle$ is the sum of two maximally connected sets which are domains relative to $K + \langle amb \rangle$, such that one of these has the simple closed curve ao_1bma for its boundary and the other has for its boundary the simple closed curve ao_2bma, where o_1 and o_2 are any points of K separating a and b.* It will be remarked that (A, E) imply (B, C) so that (A, E) and (A, B, C) are equivalent: § therefore the spaces which we are considering may properly be termed Jordan Curve Theorem spaces.

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† (J. M. T.) 6.1: it will be observed that the arcs used in the arguments of that section are all subarcs of simple closed curves.

‡ The symbol $\langle amb \rangle$ stands for the arc amb without its endpoints.

§ In these implications and equivalences which we may indicate it is understood that we have continuous curve spaces in mind.

2. 1. We shall show that if J^* is a continuous curve of S, D'' an open connected subset of $S - J^*$, and G^* a continuum of $J^* \cdot \overline{D''}$, then G^* is a continuous curve.

If D'' is not a maximally connected subset of $S - J^*$ it is contained in a J^*-domain D^*, and it is seen that $G^* \subset J^* \cdot \overline{D^*}$. If G^* is not a continuous curve, by Theorem 1 it contains a continuum G not a continuous curve such that G is a subset of a maximal cyclic curve J of J^*. Since J is a subset of J^*, $S - J$ contains a maximal connected subset D such that $D^* \subset D$; and further $G \subset J \cdot \overline{D}$. If every pair of points of G disconnects G, G is a simple closed curve. Then we may suppose that there is at least one pair of points x and y of G such that $G - (x + y)$ is connected. There is in J a simple closed curve K which contains x and y. We shall show that any other point z of G separates G between x and y. If z does not belong to the simple closed curve K, it is a point of a K-domain, M. Since D is a J-domain and z is a limit point of D, $D \subset M$. Since no point of J is a cut point, there is an arc hzk such that $\langle hzk \rangle \subset M$ and $h + k \subset K$. If h and k separate x and y on K then, by (F), $\langle hzk \rangle$ divides M into two $(k + hzk)$-domains such that one of these has the boundary $hzkxh$ and the other the boundary $hzkyh$. Then since these are simple closed curves of J, D must lie entirely within one of the subdomains of M, and either x or y fails to be a limit point of D. Then h and k cannot separate x and y on K, and there is a simple closed curve $K' = xzyz'x$ (where z' is an arbitrary point on the other xy-arc of K') of J.† Let N be that K'-domain which contains D. If any point of J is not a point of $K' + N$, $J - K'$ cannot be connected and $N \cdot J$ is an open subset of $J - K'$. Then $N \cdot J$ is the sum of components of $J - K'$ (or it is vacuous in which case G is a sub continuum of K' and therefore a continuous curve) and $J' = K' + N \cdot J$ is a continuous curve.‡ Let u be any point of $N \cdot J$ and v a point of J not in $K' + N \cdot J$. There is in J a simple closed curve (uv) and therefore an arc huk such that $\langle huk \rangle \subset N$ and $h + k \subset K'$, h and k being distinct. Then, since no point of K' can separate any two points of K' nor separate a point of K' from a point of $N \cdot J$, it is seen that J' is cyclicly connected.§ Suppose $J' - (x + y)$ is connected: then it contains an arc zz', and this has a last point h' on $\langle xzy \rangle$ and a first point k' thereafter on $\langle xz'y \rangle$ and an open subarc $\langle h'k' \rangle$ of N. Since h' and k' separate x and y on K' it is seen (by the argument above for h and k) that either x or y fails to be a limit point

† If $z \subset K$, then $K' = K$.
‡ See Remark to lemma of (J. M. T.).
§ If $J \subset K' + N \cdot J$, $J' = J$.

of D. Then it follows that $J'-(x+y)$ cannot be connected. Since G is a continuum of J', and $G-(x+y)$ is connected, $G-(x+y)$ belongs to a single component of $J'-(x+y)$ and this component plus x and y is a continuous curve J'' such that $J''-(x+y)$ is connected. If J'' contains any point of $\langle xz'y\rangle$ we are led to contradiction by the argument on h' and k'. If the point z does not separate x and y in J'', there is an arc xy of $J''-z$. This arc has a last point m on $xz-z$ (of K') and a first point n thereafter on $yz-z$. The subarc mn having its endpoints only on xzy, and no point in common with $\langle xz'y\rangle$ since $(\langle xz'y\rangle)\cdot J''=0$, has its endpoints only in common with K'. Since $mn\subset J'$, $\langle mn\rangle\subset N$. If the point m is not the point x, the points m and n separate x and z on K', and (as above) either x or z fails to be a limit point of D. Similarly, the point n cannot fail to be the point y. Then mn is an arc xy, such that $\langle xy\rangle\subset N$. Where z'' is an arbitrary point of this arc, there is a simple closed curve $xz''yzx$ of J''. But it can be shown that $x+y$ separates z and z'' in J'' precisely as it was shown that $x+y$ separated z and z' in J'. Since $J''-(x+y)$ is connected it follows that z must separate x and y in J''. Since $J''\supset G$, every point of G other than x and y must separate x and y in G, and G is an arc. Then, in every event, G is a continuous curve, and therefore G^* is a continuous curve.

3. We shall prove that if C is the sum of a finite number of disjoint continuous curves of S and x is any point of C which is on the boundary of a domain D relative to C, then x is accessible from D. For if x is not accessible from D it is readily shown † that there exists an ϵ and a sequence (x_n) of points converging to x, such that each point x_n is contained in $D\cdot U_{x\epsilon}$ ‡ while no two points of this sequence can be joined by an arc of $D\cdot U_{x\epsilon}$. Then there exists an infinite set of arcs (x_ny_n) no two of which have any common point, such that each is a subarc of D, and each arc except for its y_n-point belongs to $U_{x\epsilon}$. The set of arcs has a subsequence $(x'_ny'_n)$ with a sequential continuum of condensation T. From the connectivity im kleinen of D it follows that if a point of $T\cdot U_{x\epsilon}$ belongs to D two distinct points x'_j and x'_k are on a subarc of $D\cdot U_{x\epsilon}$. Then $T\cdot U_{x\epsilon}\subset C\cdot\overline{D}$ and since $T\subset\bar{U}_{x\epsilon}$ while C is closed, $T\subset C$. Then T belongs to one of the continuous curves of C: call this F; then T is a continuous curve (§ 2. 1) and contains an arc xy such that $xy-y\subset U_{x\epsilon}$. Let p and q be points of xy distinct from

† Compare R. L. Wilder: "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), Theorem 1, pp. 342-345.

‡ $U_{x\epsilon}$ is a neighborhood of x of diameter less than ϵ.

each other and from x and y. Let U_p and U_q be neighborhoods of p and q respectively such that $\overline{U}_p + \overline{U}_q \subset U_{x\varepsilon}$, $\overline{U}_p \cdot U_q + U_p \cdot \overline{U}_q = 0$, and $(U_p + U_q) \cdot (C - F) = 0$. By the connectivity im kleinen of F, there exist neighborhoods U^*_p and U^*_q such that any point of $F \cdot U^*_p$ and $F \cdot U^*_q$ respectively can be joined to p and q respectively by an arc of $F \cdot U_p$ and $F \cdot U_q$ respectively. There exist neighborhoods U''_p and U''_q such that any point of U''_p and U''_q can be joined to p and q respectively by arcs of U^*_p and U^*_q respectively. Since p and q are points of a sequential continu of condensation of $(x'_n y'_n)$, there is an arc $x'_k y'_k$ which has a point p' in U''_p and a point q' in U''_q. There is an arc $p'p$ of U^*_p and this has a last point d on $x'_k y'_k$ and a first point d' thereafter on F; the subarc dd' belongs, clearly, to $D \cdot U_{x\varepsilon}$. There is an arc $d'p$ of $F \cdot U_p$: this has a first point a on xy. Similarly there is an arc $cc'b$, where $cc' - c' \subset U^*_q \cdot D$, c' being a point of F and c its only point on $x'_k y'_k$, and $c'b \subset F \cdot U_q$, b being its only point on xy. There is a subarc ab of xy, and a subarc cd of $x'_k y'_k$. Then $ab + bc' + c'c + cd + dd' + d'a$ is a simple closed curve K such that $\langle c'cdd' \rangle \subset D \cdot U_{x\varepsilon}$, and $c'badd' \subset F \cdot U_{x\varepsilon}$. If any arc $x'_j y'_j$ $(j \neq k)$ has a point in common with each of the K-domains, it has a point in common with K and this is necessarily a point of $\langle c'cdd' \rangle$: then there is an arc $x'_j x'_k$ of $D \cdot U_{x\varepsilon}$. Since this contradicts the construction of the original sequence (x_n), every arc of $(x'_n y'_n)$, $(n \neq k)$, belongs to one or the other of the K-domains: let M be a K-domain which contains an infinite set $(x''_n y''_n)$ of the arcs of $(x'_n y'_n)$. Choose distinct points p'', p^*, q^*, q'', on $\langle ab \rangle$ in order: $ap''p^*q^*q''b$. It can be seen that there exist neighborhoods U_{p^*} and U_{q^*} such that:

$$U_{p^*} \cdot [p''adcbq^* + (C - F) + (S - U_{x\varepsilon}) + \overline{U}_{q^*}] = 0$$
$$= U_{q^*} \cdot [q''bcdap^* + (C - F) + (S - U_{x\varepsilon}) + \overline{U}_{p^*}].$$

Find neighborhoods U''_{p^*} and U''_{q^*} as above for p and q. Then we can construct an arc $a'd^*d''c''c^*b'$ such that a' and b' are on ab in the order: $ap''a'b'q''b$; $\langle a'd^*d''c''c^*b' \rangle \subset M \cdot U_{x\varepsilon}$, $(a'd^* - d^*) + (b'c^* - c^*) \subset F$, $\langle d^*d''c''c^* \rangle \subset D \cdot U_{x\varepsilon}$, and $d''c''$ belongs to an arc $x''_m y''_m$. Let a'' be any point of $\langle a'b' \rangle$. Since a'' as well as p'' are limit points of the sequential set $(x''_n y''_n)$ it is seen that there is an arc $x''_j y''_j$ $(j \neq m)$ which has a point in each of the subdomains of M determined by the arc $\langle a'd''c''b' \rangle$, since a'' and p'' separate a' and b' on K: property (F). Then $x''_j y''_j$ has a point in common with the arc $a'd''c''b'$ and this is necessarily a point of $d^*d''c''c^*$: there is an arc $x''_m x''_j$ in $D \cdot U_{x\varepsilon}$. This contradiction of our construction of the sequence (x_n) proves that x is accessible from D.

3. 1. We shall show that no acyclic continuous curve T of S can dis-

connect S. Let T be an acyclic continuous curve of S and D_x and D_y two distinct T-domains. Then T contains an acyclic continuous curve T' irreducible about $G = T \cdot \overline{D}_x$: the endpoints of T' are a subset of G. It is seen that D_x is a T'-domain. Let D'_y be that T'-domain which contains D_y. Then T' contains an acyclic continuous curve T'' irreducible about $G' = T' \cdot \overline{D'}_y$: the endpoints of T'' are a subset of G'. Since G and G' separate S, they contain at least two points. Let p and q be two endpoints of T''. Then they are points of T', and being limit points of D'_y which is a T'-domain they are accessible from D'_y. There is an arc $py'q$, $\langle py'q\rangle \subset D'_y$. Let $p'pqq'$ be the *maximal* arc of T' which contains pq: p' and p, as well as q' and q, are not necessarily distinct. Since p' and q' are endpoints of T' and therefore points of G, they are accessible from D_x. There is an arc $p'x'q'$, $\langle p'x'q'\rangle \subset D_x$. Let M be that domain of $K = p'x'q'qy'pp'$ which contains $\langle pq\rangle$ of T''. Then $M - \langle pq\rangle = M_1 + M_2$, where M_1 and M_2 are $(K + pq)$-domains and the boundary of M_1 is $py'qp$ while the boundary of M_2 is $pp'x'q'q$. Since the point x' of D_x does not belong to M_1 and the boundary of M_1 is a subset of $T' + D'_y$, it is seen that M_1 can contain no point of D_x. Then M_1 can contain no point of G. Suppose that M_1 contains a point z of T'. Let z^*pzz' be any maximal arc of T' which contains z and p. This arc in order from z to p has a first point z'' on pq. If the arc $z''zz'$ has any point other than z'' on pq it is seen that T' contains a simple closed curve. Then $z'zz'' - z'' \subset M_1$, since $(z'zz'' - z'') \cdot py'q = 0$; and $z' \subset M_1$: but z' is an endpoint of T' and therefore a point of G. Then M_1 contains no point of T' and therefore no point of T''. Similarly, M_2 can contain no point of D'_y and therefore no point of T''. Then $T'' \cdot (K + M) = pq$. If T'' is not the arc pq, T'' has a point k in N, the K-domain distinct from M. Let kp be the arc of T'' containing k and p: p or q is the first point which this arc has on K, and one of these points fails to be an endpoint of T''. Then T'' is the arc pq. But $S - T''$ is not connected, while the arc pq being a subset of a simple closed curve of S cannot disconnect S. The contradiction shows that $S - T$ must be connected.

3.2. Let C be a finite set of disjoint continuous curves of S such that D_x and D_y are distinct domains relative to C. Let $G = C \cdot \overline{D}_x$, and let $G' = G \cdot \overline{D}$ where D is that G-domain which contains D_y. Every maximal connected subset of G' belonging to $\overline{D}_x \cdot F$, where F is one of the continuous curves of C, is a continuous curve or a point (§ 2.1), and every point of G' being a point of G is accessible from D_x (§ 3). To show that every point of G' is also accessible from D we shall have slightly to modify the argument of § 3. If a point x of G' is not accessible from D we construct, precisely

as in § 3, the arc $ad'dcc'b$ and the simple closed curve K. Here, however, points of $c'b - b$ and of $d'a - a$ although they belong to F may also be points of D. Since a and b are points of G' they are accessible from D_x: there is an open arc $\langle azb \rangle$ of D_x. Let H be that domain of the simple closed curve $azbc'cdd'a$ which contains $\langle ab \rangle$ of G'. Then $H - \langle ab \rangle$ is the sum of two domains M and N with the boundaries $abc'cdd'a$ and $azba$ respectively. It is seen that no point of N can be a point of D, and therefore no point of N can be a point of G' (compare § 3.1). Similarly no point of M can be a point of G'. Then $G' \cdot H = G' \cdot (M + \langle ab \rangle + N) = \langle ab \rangle$. We can continue the argument of § 3, precisely as there, with the added condition on our neighborhoods U_{p^*} and U_{q^*} that they are subsets of H, since now if c^* and d^* are taken as the first points of G' on the respective arcs $c''q^*$ and $d''p^*$ they must be points of $\langle ab \rangle$, and the open arc $\langle a'd''c''b' \rangle \subset D$.

Let fgh be any arc of G'. Since f and h are accessible from D_x and from D, there is an open arc $\langle fxh \rangle$ of D_x and an open arc $\langle fyh \rangle$ of D: let K'' be the simple closed curve $fxhyf$, and let M'' be that K''-domain which contains the arc $\langle fgh \rangle$. By methods we have several times used, it follows that no point of $\langle fgh \rangle$ is a limit point of $G' - fgh$. If then G' contains no simple closed curve, the maximal connected subsets of G' are points or arcs such that if g' is an arc of G' no inner point of g' is a limit point of $G' - g'$: that is, G' is a set B described in Theorem 2. Then G' is a subset of an acyclic continuous curve V of S which is irreducibly connected about it. If x and y are points of D_x and D respectively, G' separates S between x and y. Since x and y belong to some simple closed curve of S, it is readily inferred from property (E) that $x + y$ cannot disconnect S. Moreover x and y are not points of G'. Then there is an acyclic continuous curve V' of $S - (x + y)$ which contains G' † and, accordingly, separates S between x and y. But by § 3.1 no acyclic continuous curve of S can disconnect S. Therefore G' must contain at least one simple closed curve J.

Let $J \equiv acbda$, and let M be that J-domain which contains D. Since every point of G' is accessible from D there is an arc $ay'b$, $\langle ay'b \rangle \subset M$. If D_x has any point in M, $D_x \subset M$. In this case it is readily seen that D_x belongs to one of the subdomains of M determined by the arc $\langle ay'b \rangle$ and that either c or d fails to be a limit point of D_x. Since this is not possible, $D_x \cdot M = 0$, and J separates S between D_x and D. If N is the J-domain which contains D_x, it is seen that $N \cdot D = 0$, and it follows that $G' = J$.

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† By the argument of (J. M. T.), Theorem 2, the second paragraph: apply this argument to x and y, replacing $C - x$ by $S - (x + y)$.

3.3. If x and y are two points of S, and ϵ is any preassigned positive number (we may suppose ϵ less than the distance of x to y), there is a finite set C of continuous curves of S, ϵ-separating x.† Then there are two distinct C-domains, D_x and D_y. Then it follows, by § 3.2, that there is a simple closed curve J of C which separates x and y in S. No point of J is at a distance from x greater than ϵ, since $J \subset C$. Let H be that J-domain which contains x. There is an ϵ_1 such that any point of S whose distance from x is not greater than ϵ_1 can be joined to x by an arc of H, and such that ϵ_1 is less than $\epsilon/2$. Let J_1 be a simple closed curve which separates x and y, such that no point of J_1 is at a distance from x greater than ϵ_1, and let H_1 be that J_1-domain which contains x. It is seen that $H \supset J_1 + H_1$. We can construct, inductively, a sequence of simple closed curves (J_n) converging to x such that each curve of (J_n) separates x and y and such that if H_n is that J_n-domain which contains x, then for every m, $H_m \supset J_{m+1} + H_{m+1}$. Then $\prod_{n=1}^{\infty} H_n \supset x$; if $\prod_{n=1}^{\infty} H_n \supset x'$ distinct from x, it is seen that every arc $x'y$ of S must have at least one point in common with each of the simple closed curves of (J_n) and therefore it must contain the point x to which these curves converge. Then x separates x' and y in S: but no point of S is a cutpoint. Then $\prod_{n=1}^{\infty} H_n = x$, and for any preassigned ϵ' there is a domain H_m containing x of diameter less than ϵ': J_m ϵ'-separating x in S.

It is now possible to show that S satisfies the Axioms 1-4 of R. L. Moore's $\Sigma 1$ (F. A.) as we have shown it in (J. M. T.) §§ 8 and 8.1 if S is compact and in the last paragraph of § 11 if S is locally compact. The Axioms 6 and 7 are established in (J. M. T.) on the basis of the Janiszewski-Mullikin Theorem. From the considerations of the preceding paragraphs of this paper we shall show that they are consequences of our assumptions (A, B, C). Let x and x' be any two points of a simple closed curve J of S, and let J' be a simple closed curve of S ϵ-separating x, where ϵ may be supposed less than the distance of x to x', and arbitrarily small greater than zero. Since x is accessible from M, where M is either of the J-domains in S, there is an arc px such that $px - x$ belongs to M. This has a subarc qx such that $qx - x$ has no point in common with $J' + J$. Let $C = J' + J$: $qx - x$ belongs to a C-domain D. Then D is a subset of a J-domain and, containing q, it belongs to M. Let x'' be any point of that J'-domain which contains x'. Since D has x for a limit point, D belongs to that J'-domain which contains x and since this does not contain x', x'' cannot belong to D.

† (J. M. T.) § 7.

Let $G = C \cdot \overline{D}$: then G contains a simple closed curve J'' separating q from x'' (by § 3. 2). Let D' be that J''-domain which contains q: $D' \supset D$. If any point of J belongs to D', there is an open arc of points of J in D' with distinct endpoints on J'' and this arc divides D' into two subdomains such that D can belong to at most one of them: but then there are points of J'' which fail to be limit points of D. Then, since x is a point of $\overline{D}$, $x \subset J''$. In a sufficiently small neighborhood of x, C is locally identical with J: since $J'' \subset C$, there is on J'' an arc axb of J. There is an arc ab of $J'' - x$: since $J'' \subset G \subset \overline{D} \subset \overline{M} = M + J$, $ab \subset M + J$. Since x' is not a point of $\overline{D}$, x' is not a point of J''. Then a and b separate x and x' on J, and the arc ab of $J'' - x$ is an arc of $(J + M) - (x + x')$: then it has a subarc $a'b'$ such that a' and b' are on J separating x and x', and $\langle a'b' \rangle \subset M$. Since $\langle a'b' \rangle \subset C$, and $\langle a'b' \rangle \cdot J = 0$, $\langle a'b' \rangle \subset J'$ and no point of $\langle a'b' \rangle$ is at a distance from x greater than ϵ. Then we can construct the domains of § 7. 2 (J. M. T.) and it is a ready consequence of the existence of these domains that S satisfies Axioms 6 and 7 of $\Sigma 1$.

Then by § 8. 2 of J. M. T. it follows that S if it is compact is a simple closed surface, and from the last paragraph of § 11 and the succeeding sections of Theorem 6 it follows that S if it is not compact is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected point set. We shall show, in a later paper, that the use of inversion in Theorem 6 is not essential, so that the restriction that the continuous curves there treated be imbedded in Euclidean space of finite dimension can be removed: the notions of compactness and non-compactness of this paper may replace, accordingly, the use of closed and boundedness and of closed and unboundedness of (J. M. T.). It may be remarked that the number-plane is characterised among the cylinder-trees (spaces homeomorphic with the complement on a simple closed surface of a closed and totally disconnected, possibly vacuous, point set) by its non-compactness and the property that every simple closed curve determines in it one compact domain: to characterise the number-plane, therefore, it is not necessary even in (J. M. T.) to resort to inversion, as may be seen from the last paragraph of § 11.

Then we have established the following theorems:

THEOREM 3. *In order that a continuous curve S be a cylinder-tree it is sufficient that it satisfy non-vacuously the Jordan Curve Theorem.*

THEOREM 3′. *In order that a continuous curve be a simple closed sur-*

face it is sufficient that it satisfy non-vacuously the Jordan Curve Theorem, and be compact.

THEOREM 3″. *In order that a continuous curve be a number-plane it is sufficient that it satisfy the Jordan Curve Theorem non-vacuously, that it be not compact, and that every simple closed subcurve determine in it a compact domain.*

That these conditions are moreover necessary is obvious for Theorems 3′ and 3″, and a consequence for Theorem 3 of the fact that no closed and totally disconnected subset of the domains complementary to a simple closed curve on a simple closed surface can disconnect those domains.

In these theorems, further, it is possible to replace the postulation of the Jordan Curve Theorem by the equivalent system (A, B, C) or by the system (B', C) where B' is the property that if L is an arc of S, then $S - L$ is connected. It is readily seen that (B', C) implies (A, B, C).†

4. We shall show that a continuous curve S which has the property (D) of cyclic connectivity, and also has the property: (G) if K is a simple closed curve of S, every point of K is a limit point of $S - K$, and $S - K$ is the sum of two connected sets without common point; satisfies the Jordan Curve Theorem.

Let K be any simple closed curve of S. Then $S - K = M + N$, where M and N are distinct K-domains. If the Jordan Curve Theorem is not satisfied there is a point y of K which fails to be a limit point of one of the K-domains, say N. By (D), it is clear that N has at least two distinct limit points on K. Then there is an arc xyz of K such that $x + z \subset \bar{N}$, but $\langle xyz\rangle \cdot \bar{N} = 0$. By (G), $\langle xyz\rangle \subset \bar{M}$.

4.1. We consider first the case that $\langle xy'z\rangle \cdot \bar{M} = 0$, where $xy'z$ is the other xz-arc of K: $xy'z \subset \bar{N}$. For any point y'' of $xy'z$ it is possible to construct an arc pq, $\langle pq\rangle \subset N$, $p + q \subset xpy''qz \equiv xy'z$, from which it follows that no point of $\bar{N}$ separates x and z in $\bar{N}$. Since $\bar{N}$ is a continuous curve, there is a simple closed curve $K' \equiv xy_1zy_2x \subset \bar{N}$. Let $S - K' = H + D$ in maximal connected sets: since $M \cdot K' = 0$, M belongs to one of the K'-domains, say H. Then because $\langle xyz\rangle \cdot K' = 0$ and $xyz \subset \bar{M}$, $xyz + M \subset H$ and $xyz + M = H$ because $xyz + M$ is maximally connected in $S - (x + z)$ and therefore in $S - K'$. Then $\bar{H} \cdot K' = x + z$ and in consequence $D \cdot K' = K'$.

† Compare R. L. Wilder: "A Topological Definition of a Simple Closed Surface," *Bulletin of the American Mathematical Society*, Vol. 35 (1929), p. 194, Abstract No. 8.

It is seen that $\bar{D}-(x+z)$ is connected: it contains an arc y_1y_2 with a last point, which we may suppose to be y_1, on $\langle xy_1z\rangle$ and a first point y_2 thereafter on $\langle xy_2z\rangle$: it is understood that we are not here supposing accessibility, the points y_1 and y_2 having been in no previous way particularized. Then $\langle y_1y_2\rangle \subset D$. From the relation $S-zyxy_2y_1x = M + xy_1 + y_2z + (D - \langle y_1y_2\rangle)$ it is seen that the bracketed point set must be connected: else the simple closed curve $zyxy_2y_1x$ fails to disconnect S into not more than two domains. There is an arc pq, $p \subset \langle xy_1\rangle$, $q \subset \langle zy_2\rangle$, $\langle pq\rangle \subset D - \langle y_1y_2\rangle$. From the relation $S - zy_1y_2z = H + \langle y_1xy_2\rangle + (D - \langle y_1y_2\rangle)$, it is seen that $D - \langle y_1y_2\rangle$ cannot be connected and has a maximally connected subset A_z which has no limit point in $\langle y_1xy_2\rangle$: otherwise zy_1y_2z fails to separate S. If any other component (maximal connected subset) of $D - \langle y_1y_2\rangle$ fails to have a limit point in y_1xy_2, then $S - zy_1y_2z$ has at least three components. If A_z has no limit point in $\langle y_1z\rangle$, then $S - zy_2y_1xyz$ has at least three components: one of these is A_z, the other is M, and a third exists containing for example $\langle xy_2\rangle$. From $S - xy_1y_2x = H + \langle y_1zy_2\rangle + (D - \langle y_1y_2\rangle)$ it is seen that $D - \langle y_1y_2\rangle$ has a component A_x such that $\bar{A}_x \cdot \langle y_1zy_2\rangle = 0$, and such that this is not true for any other component: then $A_x \neq A_z$. If $\bar{A}_x \cdot y_2x = 0$, $S - xy_1y_2zyx$ has at least three components.

Since $(pq)\cdot(\langle y_1xy_2\rangle) \neq 0 \neq (\langle y_1zy_2\rangle)\cdot(pq)$, it follows that $\langle pq\rangle \subset A'$, a component of $D - \langle y_1y_2\rangle$ distinct from A_x and from A_z. There is in D an arc joining a point of $\langle pq\rangle$ to a point of $\langle y_1y_2\rangle$: this has a last point h on $\langle pq\rangle$ and a first point k thereafter on $\langle y_1y_2\rangle$. It is seen that $hk - k \subset A'$. Then $\bar{A}_x \cdot \langle y_1k\rangle \neq 0$; otherwise $S - xy_1zqhky_2x$ has at least three components (one of these is M, and another $\bar{A}_x$: a third contains, for example, $\langle y_2q\rangle$). If $\bar{A}_z \cdot \langle y_2k\rangle = 0$, $S - zy_1khpxy_2z$ has at least three components. Knowing that $\bar{A}_x \cdot \langle y_2x\rangle \neq 0 \neq \bar{A}_z \cdot \langle y_1z\rangle$, we can deduce the existence of an arc rs, $\langle rs\rangle \subset A_x$, $r \subset \langle y_2x\rangle$, $s \subset \langle y_1k\rangle$, and an arc $r's'$, $\langle r's'\rangle \bar{A}_z$, $r' \subset \langle y_1z\rangle$, $s' \subset \langle y_2k\rangle$.

Then $\bar{A}_x \cdot \langle y_2s'\rangle \neq 0$, or $S - s'y_1xy_2zr's'$ has at least three components, and $\bar{A}_z \cdot \langle sk\rangle \neq 0$, or $S - sy_1zy_2khpxrs$ has at least three components. Then there exist two arcs ad and bc such that $\langle ad\rangle \subset A_x$ and $\langle bc\rangle \subset A_z$, and a, b, c, and d are points on $\langle y_1y_2\rangle$ in the order: $y_1abkcdy_2$. There is an arc $r''x'$, $\langle r''x'\rangle \subset A_x$, $r'' \subset \langle y_2x\rangle$, and $x' \subset \langle ad\rangle$. There is an arc r^*z', $\langle r^*z'\rangle \subset A_z$, $r^* \subset \langle y_1z\rangle$, and $z' \subset \langle cb\rangle$. Let K'' be the simple closed curve $ax'dcz'ba$ and let U be that K''-domain which contains H: $U \supset H + K' + \langle y_1a\rangle + \langle bkc\rangle + \langle dy_2\rangle$. If V is the other K''-domain, $V \subset D - y_1y_2$. If $\bar{V}\cdot(ax'd + cz'b) = 0$, $\bar{V}\cdot K'' \subset (ab + cd)$. Then $S - bax'dcz'r^*zy_2xphkb$ has at least three components. Since V contains no point of y_1y_2 it cannot

contain points of both A_x and A_z, and in consequence a point of A_x and of A_z cannot both be limit points of V. Then from the relation above it follows also that no point of $\langle ax'b\rangle$ can be a limit point of V. But in this case $S - abz'cdx'r''xyzy_1a$ has at least three components.

Then we have shown that for no simple closed curve K of S can one of the K-domains have at most two limit points on K, and further that if there is an arc xyz of K such that $\langle xyz\rangle \cdot \bar{N} = 0$, $x + z \subset \bar{N}$, then at least one point y' on the other xz-arc of K is in M, where M and N are the respective K-domains.

4. 2. Returning to § 4, we can say that $\langle xy'z\rangle \cdot \bar{M} \supset y'$, and that $\langle xy'z\rangle$ contains moreover a point y^* of $\bar{N}$. Let us suppose that both x and z are accessible from N. In this case there is an arc $xy''z$, $\langle xy''z\rangle \subset N$. From the relation $S - xy'zy''z = M + \langle xyz\rangle + (N - \langle xy''z\rangle)$, it is seen that $(N - \langle xy''z\rangle)$ must be connected: for $M + \langle xyz\rangle$ is connected and no point of it is a limit point of $(N - \langle xy''z\rangle)$, nor is any point of $(N - \langle xy''z\rangle)$ a limit point of $M + \langle xyz\rangle$. But from the relation $S - xyzy''z = M + \langle xy'z\rangle + (N - \langle xy''z\rangle)$ it is clear that $(N - \langle xy''z\rangle)$ cannot be connected: for $\langle xy'z\rangle$ contains limit points both of M and of N, and if $N - \langle xy''z\rangle$ is connected, $\langle xy'z\rangle$ has a limit point of it also and the right side of the above relation is a connected point set. Then if the Jordan Curve Theorem is to fail to be true in S, it follows that x and z cannot both be accessible from N.

4. 3. Suppose then that one of these points, say x, is accessible from N. Then the other is not. But then it can be shown, as in § 3, that z is a point of a continu of condensation T such that every point of T is a limit point of N. Since T is a subset of K, and moreover, since $\langle xyz\rangle \cdot N = 0$, T is a subarc of $xy'z$: T is an arc tz. Then if t' is any point whatever of $\langle tz\rangle$, there is a point t'' of $\langle t'z\rangle$ and an arc xt'' such that $\langle xt''\rangle \subset N$.

We may assume that the point t'' separates y' and z on $xy'z$. From $S - xyt''x = M + \langle xy't''\rangle + (N - \langle xt''\rangle)$ it is seen that $(N - \langle xt''\rangle)$ cannot be connected, and has a component which fails to have any point of $\langle xy't''\rangle$ for limit point. It is conceivable, however, that this domain fails, also, to have any limit point on $t''z - tt''$. We shall show that if z is to be not accessible from N, then for some arc xt'' this must in fact happen. More precisely, for any choice of t', the arc xt'' exists, t'' separating t' and z.

Let xp_1 be an arc such that $\langle xp_1\rangle \subset N$, and $p_1 \subset \langle tz\rangle \cdot y'z$; let D_1 be that domain of $N - \langle xp_1\rangle$ for which $\bar{D}_1 \cdot \langle xy'p_1\rangle = 0$. If, then, $\bar{D}_1 \cdot (zp_1 - p_1) \neq 0$, let z_1 be the point such that $z_1 \subset \bar{D}_1 \cdot (zp_1 - p_1) \subset p_1z_1$. If z_1 is accessible from D_1, there is an arc xx_1z_1 such that $xx_1 \subset xp_1$,

$\langle x_1 z_1 \rangle \subset D_1$. Then $N - \langle x x_1 z_1 \rangle$ has a component $D' \subset D_1$ such that $\bar{D}' \cdot \langle x p_1 z_1 \rangle = 0$, but $\bar{D}' \cdot \langle z_1 z \rangle \neq 0$. This contradicts our choice of z_1 unless $z_1 = z$; and this is not possible since z is supposed not accessible. Therefore z_1 is not accessible, and is seen to be a limit point of $\bar{D}_1 \cdot \langle p_1 z_1 \rangle$. We shall choose an arc $x_2 p_2$, $x_2 \subset x p_1$ and $p_2 \subset p_1 z$, and $\langle x_2 p_2 \rangle \subset D_1$, subject to the following conditions: (1) p_2 is a point of $\bar{D}_1 \cdot \langle p_1 z_1 \rangle$ such that the diameter of the arc $p_2 z_1$ is less than $\epsilon/2$, where ϵ is the diameter of the arc $p_1 z_1$; (2) if for some choice of p_2 subject to (1) and for some choice of x_2 on $\bar{D}_1 \cdot \langle x p_1 \rangle$ (it is seen that this cannot be a vacuous set, since N is connected) there is an arc $x_2 p_2$, $\langle x_2 p_2 \rangle \subset D_1$, whose diameter is d_2 then that arc $x_2 p_2$ which we choose shall have a diameter not exceeding $d_2 + \epsilon/2$. We may express (2) in this form, that we choose a particular arc from among the set (non-vacuous) of available arcs so that its diameter does not exceed the lower limit of the diameters of the available arcs by more than $\frac{1}{2}\epsilon$: otherwise the choice of our arc is free (except as subject to (1)).

Then $N - \langle x x_2 p_2 \rangle$ has a domain $D_2 \subset D_1$ such that $\bar{D}_2 \cdot \langle x p_1 p_2 \rangle = 0$, but $\bar{D}_2 \cdot \langle p_2 z \rangle \neq 0$. There is a point $z_2 \subset \bar{D}_2 \cdot p_2 z \subset p_2 z_2$. Let $x_3 p_3$ be an arc, $x_3 \subset x p_2$, $p_3 \subset p_2 z_2$, $x_3 p_3 \subset D_2$, such that: (1) the diameter of the arc $p_3 z_2$ is less than $\epsilon/4$; (2) the diameter of the arc $x_3 p_3$ is at most $d_3 + \epsilon/4$, where d_3 is the lower limit of the diameters of the arcs $(x_3 p_3)$ which satisfy (1). We construct, inductively, for every n, an arc $x x_n p_n$, a domain D_n, and a point z_n. From the fact that z_{m+1} either precedes z_m on $xy'z$, or is identical with it, it follows that the set of points (z_n) has a sequential limit z' which either precedes every point of (z_n) or is identical with all but a finite number of them. It follows from (1) that z' is also the sequential limit point of the set of points (p_n).

Suppose that infinitely many of the arcs $(p_n x_n)$ are of diameter greater than some ϵ'. Since S is locally compact, we may suppose ϵ' such that $U_{z'\epsilon'}$ is contained in a compact subset of S. Then if $(p'_n x'_n)$ is the set of arcs of $(p_n x_n)$ each of diameter greater than ϵ', where we suppose this subsequence such that for every n, $p'_n \subset U_{z'\epsilon'}$, there is on each arc in order from p'_n a first point q'_n which is not contained in $U_{z'\epsilon'}$. The set of arcs $(p'_n q'_n)$ has a subsequence $(p''_n q''_n)$ with a sequential continuum of condensation T. Since all but a finite number of the arcs of $(p''_n q''_n)$ are contained in $\bar{D}_m$ for every m, it is seen that no point of $\langle x p_{m-1} p_m \rangle + \langle z_m z \rangle$ can be a limit point of T. If then x is not contained in $U_{z'\epsilon'}$, T has no point in common with $xy'z$ excepting z'. Let t be a point of T distinct from z'. Then $t \subset \bar{D}_m$, for every m: we choose t moreover so that it is contained in $U_{z'\epsilon'/2}$. There is a k such that $p_j x_j$, $j > k$, is of diameter not

exceeding $d_j + \epsilon'/4$, where d_j is the lower limit of the diameters of the "available arcs $(p_j x_j)$".

Case 1: suppose $t \subset p''_i q''_i$, $i > k$. There is a neighborhood U_t of t such that any point of U_t may be joined to t by an arc of $U_{z'\epsilon'/2}$. Let t' be any point of an arc $p''_s q''_s$, $s > i$, such that $t' \subset U_t$. There is an arc $t't$ in $U_{z'\,1/2\epsilon'}$: this has a last point t'' on the arc $\langle p''_s q''_s \rangle$, and a first point t^* thereafter on $\langle p''_i q''_i \rangle$. Since $p_s'' q''_s - p''_s \subset D''_i$, where D''_i is that one of the domains D_n which corresponds to the renumbered arc $p''_i q''_i$, it follows that $p''_s t'' t^*$ is contained, except for its endpoints, in D''_i. Say the domain D''_i is the domain D_r, that is, in the set (D_n). Then it is seen that the arc $p''_s t'' t^*$ corresponds to an arc $x_{r+1} p_{r+1}$ of diameter less than $\epsilon'/2$: therefore the arc $x_{r+1} p_{r+1}$ which was actually chosen should not have been of diameter greater than $3\epsilon'/4$. In this case we may conclude that not more than a finite number of the arcs $x_n p_n$ are of diameter greater than a preassigned ϵ.

Case 2: t is not a point of any arc $x p_k$. Then t is contained in D_n for every n. Let U_t be a neighborhood of t as above. Again, if $p''_s q''_s$ and $p''_i q''_i$ are arcs of $(p''_n q''_n)$ which have points t' and s' respectively in U_t, and $i > s > k$ (as above), we can obtain the contradiction of Case 1.

Then we may conclude that the arcs $x_n p_n$ have been so constructed that not more than a finite number of them are of diameter greater than any preassigned ϵ. From this, and the construction of the arcs $x p_n$, it is readily seen that the sum of all of the arcs $(x p_n)$ is an acyclic continuous curve V whose points on $xy'z$ consist of x, z', and the points of (p_n). Then there is an arc xz' in this curve V which has no point other than x and z' in common with $xy'z$, because the points of (p_n) are endpoints of V. This arc xz', moreover, is contained in $\bar{D}_n$ for every n. There is a component of $N - \langle xz' \rangle$ which has no limit point in $\langle xy'z' \rangle$ and a limit point z'' in $\langle z'z \rangle$. It is seen that z'' is a limit point of every D_n. But for some n, points of z_n must precede z''. Since this is impossible because the point z_n is the last limit point which D_n has on $xy'z$, it follows that $z'' = z' = z$, and z is accessible. Therefore, our assertion is proved, that if x is accessible and z is not, there is for every point p' of $xy'z$, an arc xp such that $p \subset \langle p'z \rangle$, $xp \subset N$, and that component of $N - \langle xp \rangle$ which has no limit point on $\langle xy'p \rangle$ has no limit point also on $pz - p$: i. e. xp separates S.

4. 3. If neither x nor z is accessible, it can be shown by an inconsiderable modification of the above argument (in which it may be necessary to proceed first towards x or z, or towards x and z simultaneously) that if

p' and q' are two points of $xy'z$, there exists an arc pq, such that $pq \subset N$, and p and q lie on $xy'z$ in the order $xpp'y'qq'z$, and $S - pq$ is not connected: there is a domain $D_{pq} \subset N$.

4.4. Moreover if x, say, is not accessible then x cannot be a limit point of points of $\bar{M} \cdot xy'z$. For since x is not accessible there is an arc pq, where p may be the point z, such that $\langle pq \rangle \subset N$, $p + q \subset xy'z - z$, and $S - \langle pq \rangle$ contains a domain D which is a component of $N - \langle pq \rangle$. Let m' be a point of $\bar{M} \cdot \langle qx \rangle$, and let m be a point of $\bar{N} \cdot \langle m'x \rangle$ (such points exist, since x is not accessible). Then m is a limit point of a component of $N - \langle pq \rangle$ which has also a limit point m'' on the arc $\langle py'q \rangle$ of $xy'z$: i. e. there is only one domain of $N - \langle pq \rangle$ which fails to have a limit point on $py'q$—and in this case we have assumed that this domain has no limit point on $\langle qx \rangle$. Then there is an arc mm'' (we may suppose these points so chosen that they are accessible) such that $\langle mm'' \rangle \subset N$. If now mm'' has a domain D', i. e. if there is more than one component in $S - \langle mm'' \rangle$, or differently expressed, if $N - \langle mm'' \rangle$ has a component D' which, failing to have any limit point on the subarc $\langle m''m \rangle$ of $xy'z$, has no limit point also on either of the arcs $\langle zm'' \rangle$ or $\langle mx \rangle$, then the simple closed curve $pqmm''p$, where $\langle pq \rangle$ is the arc of N and mm'' as well as qm and $m''p$ are subarcs of $xy'z$, has at least three components. Then that domain D' of $N - \langle mm'' \rangle$ which has no limit point in $\langle m''m \rangle$ of $xy'z$ (such a domain exists from the existence of the point m' on $\langle mm'' \rangle$) must have a limit point on $\langle zm'' \rangle$ or $\langle xm \rangle$, and arguing within this domain we can show that there is no point which is either a last accessible point, or limit point z' of last accessible points, so that the accessibility of x must result.

4.5. We shall construct in M a simple closed curve which fails to separate S: the construction will resemble in numerous details the procedure of § 4.1.

Since every point of $\langle xy \rangle$ as well as of $\langle yz \rangle$ on K (see § 4) is a limit point of M, it is readily seen that there exists an arc hk, $\langle hk \rangle \subset M$, $h \subset \langle xy \rangle$, $k \subset \langle yz \rangle$. Then $S - hxy'zkh = N + \langle hyk \rangle + (M - \langle hk \rangle)$, and it follows that $\langle hyk \rangle + (M - \langle hk \rangle)$ is connected: in fact it is an $(hxy'zkh)$-domain. Since y' is a limit point of M, it is a limit point of $(M - \langle hk \rangle)$, therefore of $\langle hyk \rangle + (M - \langle hk \rangle)$. There exists an arc st, $s \subset \langle hyk \rangle$, $t \subset \langle xy'z \rangle$, $st \subset (M - \langle hk \rangle)$: $(st) \cdot (hk) = 0$. Since $S - stxs = N + \langle szt \rangle + (M - \langle st \rangle)$, there is a component A of $(M - st)$ which has no limit point in $\langle szt \rangle$, and A is the only such component: for if no such component exists, $S - stxs$ is connected, and if more than one then there are in $S - stxs$

at least three domains. If $\bar{A} \subset A + st$, let pq be an arc such that $\langle pq\rangle \subset N$, $p \subset \langle xt\rangle$, $q \subset \langle tz\rangle$. If, now, $N - \langle pq\rangle$ has any component D such that $\bar{D} \subset D + pq$, the simple closed curve $stpqzs$, where tp and qzs are arcs of K, has at least three domains. Then no such arc pq exists, and either x or z must be accessible. If now x, say, is accessible let xq be an arc such that $\langle xq\rangle \subset N$, $q \subset \langle tz\rangle$, and $N - \langle xq\rangle$ has a domain D' such that $\bar{D}' \subset D' + xq$: then $stxqzs$ has three domains, and z also is accessible. But we have previously shown that x and z cannot both be accessible. Therefore it follows that A has at least one limit point in $\langle sxt\rangle$.

Since $S - stzs = N + \langle sxt\rangle + (M - \langle st\rangle)$, it is seen that $N - \langle st\rangle$ has a component B, and only one, such that $\bar{B} \cdot \langle sxt\rangle = 0$; it follows the argument above that $B \cdot \langle szt\rangle \neq 0$, and it is clear that B and A are distinct. Moreover, since $hk \cdot st = 0$ and therefore $\langle hk\rangle \subset (M - \langle st\rangle)$ and has a limit point on both $\langle sxt\rangle$ and $\langle szt\rangle$, it is seen that $M - \langle st\rangle$ has at least one other component C. We shall show, however, that $M - \langle st\rangle$ cannot have more than the two components A and B. Suppose that no point of $xs - s$ is a limit point of A. Then there is on $\langle xt\rangle$ a point x'' such that $x''x \cdot \bar{A} = 0$. Let pq be an arc, $\langle pq\rangle \subset N$, $p \subset x''x$, $q \subset tz - t$. Since $\bar{A} \subset A + \langle stx''\rangle$, it is seen as in the foregoing paragraph that the component of $N - \langle pq\rangle$ which has no limit point in $\langle ptq\rangle$ cannot fail to have limit points in $xp - p$ or in $zq - q$, and that in consequence both x and z must be accessible from N (for in the arc pq, p may be x or q may be z). Then $(xs - s) \cdot \bar{A} \neq 0$: suppose, however, that $\langle xs\rangle \cdot \bar{A} = 0$. If x is not a limit point of points of $\bar{A} \cdot \langle tx\rangle$, it readily follows that x must be accessible from A. If x is a limit point of points of $\bar{A} \cdot \langle tx\rangle$ it follows that x is accessible from N (§ 4.4) and then that z is accessible, since p may be x in the arc, above, pq. Therefore either x is accessible from A, or there is at least one point in $\bar{A} \cdot \langle sx\rangle$, and consequently a point $x' \subset \bar{A} \cdot \langle sx\rangle$ such that x' *is* accessible from A. It will be immaterial to the sequel whether the point x' or x is accessible from A, and therefore we shall write x' with the understanding that x' is a point of $xs - s$, and may be x.

By an entirely analogous argument there is a point z' on $zs - s$ such that z' is accessible from B. Let a be any point of $\bar{A} \cdot \langle st\rangle$ such that there is an arc ax', $\langle ax'\rangle \subset A$: the existence of such a point readily follows from this, that $\bar{A} \cdot \langle st\rangle$ cannot be vacuous. Suppose that $\bar{B} \cdot \langle at\rangle = 0$: the simple closed curve $ax'tz'sa$ has at least three domains, one of them N, the other B, and a third containing $\langle sx'\rangle$, for example. Then there is a point b of $\bar{B} \cdot \langle at\rangle$ such that there is an arc bz', $\langle bz'\rangle \subset B$. Let H be that $(bz'tx'sab)$-domain which does not contain N. Since $S - bz'tx'sab = N + \langle sz'\rangle + (M - \langle sbz'\rangle)$, H must contain the set $\langle sz'\rangle + (M - \langle sbz'\rangle)$. There is

then an arc fg of H, $fg \cdot \langle bt \rangle = f$ and $fg \cdot \langle sz' \rangle = g$. Then $\langle fg \rangle$ belongs to a component of $M - \langle st \rangle$, and this component cannot be A.

Case 1: Suppose $\langle fg \rangle \subset B$. If $\bar{A}$ has no limit point in $\langle ft \rangle$, the simple closed curve $fgz'tx'sf$ has at least three components: then there is an arc $a'ka$, $\langle a'ka \rangle \subset A$, $a' \subset \langle ft \rangle$, and $a'k \cdot ax' = k$. Let H' be that $btz't$-domain which does not contain N. Then H' does not contain the arc $txsz'$, or the arc sb, or the arc $z'f$: $H' \subset M - \langle st \rangle$ and belongs to a component of $M - \langle st \rangle$ which has no limit point in $\langle sxt \rangle$; and $H' \subset B - \langle bz' \rangle$. Since g is a limit point of B, therefore of $B - \langle bz' \rangle$, it follows that H' is a proper subset of $B - \langle bz' \rangle$ and has at least one limit point in $\langle bz' \rangle$. Moreover, if $H' \cdot \langle a't \rangle = 0$, the simple closed curve $a'kx'sz'ba'$ has at least three components, H', N, and a third which contains, for example, $\langle x'sz' \rangle$. Then there is an arc $b'h$, $\langle b'h \rangle \subset H'$, $b' \subset \langle a't \rangle$, and $h \subset \langle bz' \rangle$. Let H'' be that $aka'b'hba$-domain which does not contain N. Then H'' contains no point of $sxtzs + sa + gf + bfa' + b't$: being a subset of $M - \langle st \rangle$ it belongs to a component of $M - \langle st \rangle$. Since every component of $M - \langle st \rangle$ other than A and B is maximally connected in $S - aka'b'hba$ and has at least one limit point in $sxtzs$, no component other than A and B can have any point in H''. Then $H'' \subset A + B$, and since $H'' \cdot \langle st \rangle = 0$, H'' belongs to A or H'' belongs to B. In the first case the simple closed curve $baka'b'hz'tx'sgfb$ has at least three components, and in the second the simple closed curve $bhb'a'kx'tz'gfb$.

Case 2: $\langle fg \rangle$ does not belong to B. Then $\langle fg \rangle$ is contained in a component C' of $M - \langle st \rangle$, distinct from B and from A. As in Case 1, there is an arc $a'ka$, $\langle a'ka \rangle \subset A$, $a' \subset \langle ft \rangle$, $a'k \cdot x'a = k$. If B has no limit point in $\langle a't \rangle$, the simple closed curve $a'kx'tz'sa'$ has at least three domains. Since $\langle bz' \rangle \subset B$, there is an arc $b'h$, $\langle b'h \rangle \subset B$, $b' \subset \langle a't \rangle$, $h \subset \langle bz' \rangle$. The simple closed curve $aka'b'hba$ has no point in common with $\langle fg \rangle$ since this belongs to C', and we arrive at a contradiction as in Case 1.

Then we have shown that S satisfies the Jordan Curve Theorem, and S is a cylinder tree. We have the

THEOREM 4. *A necessary and sufficient condition that a continuous curve be a cylinder tree is that it be cyclicly connected and satisfy the property* (G): *if K is a simple closed curve of S, then every point of K is a limit point of $S - K$ and $S - K$ is the sum of precisely two components.*

5. We shall show that the Theorem 4 remains true if the condition of

cyclic connectivity be replaced by the weaker condition that S contains at least one simple closed curve (property A).

For if S contains a simple closed curve, it contains a maximal cyclic curve J. If any two points x and y of J are cut-points of S, and K is a simple closed curve of J containing x and y, it is seen that $S - K$ has at least three components. Therefore J has at most a single cut-point x of S. On our definition of a continuous curve, $J - x$ is a continuous curve J'. If any point y of J' is a cut-point of J', and if K is a simple closed curve of J containing x and y, it is readily seen that $S - K$ contains at least three distinct components. Therefore J' is cyclicly connected. If K' is any simple closed curve of J', every point of K' being a limit point of $S - K'$, but not a limit point of $S - J'$, is necessarily a limit point of $J' - K'$; and, moreover, if $J' - K'$ consists of more than two components it follows that $S' - K'$ consists of more than two components. Then J' has the properties (D) and (G) and by the theorem above is a cylinder tree.

Let S' be the simple closed surface such that J' is homeomorphic with $S' - B$, where B is a closed and totally disconnected point set. Since x of J is a limit point of J', there is a sequence of points of J' converging to x, and the corresponding points on S' converge to a point x' of S' which is necessarily a point x' of B. Moreover since J is locally compact, in a neighborhood of x every infinite set of points of J' has a limit point—which may be x. Then in a neighborhood of x' on S', every infinite set of points of $S' - B$ has a limit point which may be x', but is not a point of $B - x'$. If the homeomorphism be extended to x and x', that is if x and x' be defined as corresponding points in this transformation, it follows that there exists a homeomorphism between J and $S' - (B - x')$, and $B - x'$ is totally disconnected and closed. That is, J is a cylinder-tree. If then, K'' is a simple closed curve of $S' - (B - x')$ which contains x', it separates $S' - (B - x')$ into two distinct components, and if K^* is the corresponding simple closed curve of J, containing x, $J - K^*$ is the sum of two distinct components. Then it follows that x cannot be a cut-point of S, or $S - K^*$ contains at least three components. Then S is also cyclicly connected, and is a cylinder tree.

THEOREM 5. *A necessary and sufficient condition that a continuous curve be a cylinder-tree is that it contain at least one simple closed curve and have the property G.*

6. Suppose now that S is any space satisfying the following Axioms: Axiom 1, 4, and 5 of $\Sigma 1$,†

† F. A.

Axiom 2′: a region is a connected set of points, not a single point.

Axiom 3′: if x is a point of S, $S - x$ is connected.

Axiom 8′: If K is a simple closed curve of S, every point of K is a limit point of $S - K$, and $S - K$ is the sum of two components at least one of which is compact.

From the work of E. W. Chittenden † it follows that S, satisfying Axioms 1 and 2′ is a metric separable space. From Axiom 3′ it follows that S is connected, and from Axiom 4 that S is locally compact. From Axioms 1 and 2′ it follows, readily, that S is connected im kleinen; with Axiom 3′ that S is cyclicly connected. Axiom 5 asserts that S is not compact, Axiom 8′ that every simple closed curve of S determines a compact domain, and that S has moreover the property G. Then S is a cylinder tree, and moreover a number-plane.

It is possible, in view of Theorem 5, to replace Axiom 3′ by either of the following Axioms:

Axiom 3″ S contains at least one simple closed curve.

Axiom 3* if m is any arc of S, then at least one point of m fails to disconnect S.

We have, finally, the theorem:

THEOREM 6. *A necessary and sufficient condition that a space S be a number-plane is that it satisfy the Axioms,* 1, 2′, 3′, 4, 5, *and* 8′.

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† "On the Metrization Problem and Related Problems in the Theory of Abstract Sets," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 22, No. 8.

The Linear Element of a Riemannian V_n in Terms of the Christoffel Symbols of the Second Kind.*

By W. C. Graustein.

1. *Introduction.* The problem of determining conditions necessary and sufficient that there exist a nonsingular quadratic differential form in n variables $x^1, x^2, \cdots, x^n$:

$$(1) \qquad ds^2 = g_{ij}dx^i dx^j, \qquad g_{ij} = g_{ji}, \qquad g \equiv |g_{ij}| \neq 0,$$

which has as its Christoffel symbols $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ prescribed functions, Γ^i_{jk}, has been successfully treated in all its generality.† The solution obtained, though ostensibly the best possible for the unrestricted problem, is involved and fails to furnish conditions which can readily be expressed explicitly in terms of the given functions Γ^i_{jk}.

It is the purpose of this paper to show that if $n = 2$, or if (1) is restricted to be the linear element of a Riemannian space of constant Riemannian curvature or, more generally, of an Einstein space, conditions of a very simple nature, expressed directly in terms of the given functions Γ^i_{jk}, can be found.

The general analytic problem has various geometric interpretations. The conditions sought are the conditions necessary and sufficient (a) that the functions Γ^i_{jk} be the Christoffel symbols of the second kind of a Riemannian space V_n; (b) that the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \qquad (i = 1, 2, \cdots, n)$$

be those of the geodesics of a Riemannian space V_n and s be the arc of an arbitrary geodesic; (c) that the space V_n with the coefficients of connection Γ^i_{jk} be a Riemannian space.

We shall find it convenient to formulate our treatment and results in terms of the geometric language of the last of these interpretations.

2. *Preliminary Considerations.* Since $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ is symmetric in j and k, necessary conditions on the functions Γ^i_{jk} are

$$\Gamma^i_{jk} = \Gamma^i_{kj}.$$

* Presented to the Society, December 28, 1926.

† Cf. Eisenhart, "Non-Riemannian Geometry," *American Mathematical Society Colloquium Publications*, Vol. VIII, § 29.

We assume henceforth that these conditions are always satisfied, i. e. that the connection Γ^i_{jk} is always symmetric.

The curvature tensor of the space with the symmetric connection Γ^i_{jk}, namely

$$B^i_{jkl} = (\partial\Gamma^i_{jl}/\partial x^k) - (\partial\Gamma^i_{jk}/\partial x^l) + \Gamma^i_{hk}\Gamma^h_{jl} - \Gamma^i_{hl}\Gamma^h_{jk},$$

and its contraction, $B_{jk} = B^i_{jki}$, are the generalizations respectively of the Riemannian curvature tensor R^i_{jkl} and the Ricci tensor R_{jk} of a Riemannian space. Since the tensor R_{ij} is symmetric, necessary conditions on Γ^i_{jk} are that the tensor B_{ij} be symmetric.

When $B_{ij} = B_{ji}$, the projective curvature tensor of Weyl has the form

$$W^i_{jkl} = B^i_{jkl} - [1/(n-1)]\ (\delta_l{}^i B_{jk} - \delta_k{}^i B_{jl}),$$

and the relations

$$B_{ih}W^h_{jkl} + B_{hj}W^h_{ikl} = B_{ih}B^h_{jkl} + B_{hj}B^h_{ikl}, \tag{2}$$

which will prove of use later, are readily established.

3. *A General Theorem.* As the point of departure for the consideration of the general problem, we employ the well known proposition.*

LEMMA 1. *A necessary and sufficient condition that the functions Γ^i_{jk}, symmetric in j and k, be the Christoffel symbols of the second kind of the differential form* (1) *is that*

$$g_{ij,k} = 0, \tag{3}$$

where the covariant differentiation is with respect to Γ^i_{jk}.

If for g_{ij} we set

$$g_{ij} = e^{-\phi}a_{ij},$$

where $\phi(x^1, x^2, \cdots, x^n)$ is an invariant function, equations (3) become

$$a_{ij,k} = \phi_{,k}a_{ij}, \tag{4}$$

where $\phi_{,k}$ is the gradient of the function ϕ.

If a tensor a_{ij} satisfies equations of the form (4), it obviously satisfies equations of the form

$$a_{ij,k} = \phi_k a_{ij}, \tag{5}$$

where ϕ_k is a covariant vector. It also satisfies the equations

$$a_{ih}B^h_{jkl} + a_{hj}B^h_{ikl} = 0, \tag{6}$$

resulting from the conditions of compatibility,

$$a_{ij,kl} - a_{ij,lk} = a_{ih}B^h_{jkl} + a_{hj}B^h_{ikl},$$

of the equations (4).

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* Cf. Eisenhart, *loc. cit.*, § 29.

Conversely, if a tensor a_{ij} satisfies equations of the form (5) and also equations (6), it satisfies equations of the form (4). For, if a_{ij} satisfies (5), it satisfies the equations of compatibility of (5), which reduce to

$$a_{ij}(\phi_{k,l} - \phi_{l,k}) = 0.$$

Hence, if $a_{ij} \neq 0$, ϕ_k is the gradient of a function $\phi: \phi_k = \phi_{,k}$, and a_{ij} satisfies (4). On the other hand, a tensor a_{ij} all of whose components are zero surely satisfies (4).

Since equations (5) and (6) are equivalent to equations (4) and hence to equations (3), we conclude:

THEOREM 1. *A necessary and sufficient condition that there exist a tensor $g_{ij} \neq 0$ satisfying equations (3) is that there exist a tensor $a_{ij} \neq 0$ which satisfies the equations*

$$a_{ij,k} = \phi_k a_{ij}, \qquad a_{ih}B^h_{jkl} + a_{hj}B^h_{ikl} = 0, \tag{7}$$

where ϕ_k is a covariant vector. Then ϕ_k is the gradient of a function ϕ, and $g_{ij} = ce^{-\phi}a_{ij}$, where c is an arbitrary constant, satisfies (3).

We have also:

LEMMA 2. *A necessary and sufficient condition that equations (3) possess a solution of the form*

$$g_{ij} = \rho a_{ij},$$

where a_{ij} is a given tensor, $\neq 0$, and ρ an invariant function, $\neq 0$, is that a_{ij} satisfy equations (7). Then ϕ_k is the gradient of a function ϕ and

$$g_{ij} = ce^{-\phi}a_{ij}, \qquad c \neq 0,$$

is the general solution of (3) of the prescribed form.

The first set of equations (7), i. e. equations (5), may be replaced by the equivalent equations

$$a_{rs}a_{ij,k} - a_{ij}a_{rs,k} = 0.$$

In other words, equations (5) are to be thought of as demanding merely that $a_{ij,k}$ be equal to the product of a_{ij} and *some* covariant vector ϕ_k—not one which is prescribed. The same significance is to be ascribed to every later set of equations of the form (5).

If a_{ij} satisfies equations (7), $\frac{1}{2}(a_{ij} + a_{ji})$ satisfies these equations, and $g_{ij} = \frac{1}{2}ce^{-\phi}(a_{ij} + a_{ji})$ is a symmetric tensor which satisfies equations (3). If, then, $g \neq 0$, the space with the given symmetric connection Γ^i_{jk} is Riemannian.

4. *The Case $n = 2$.* It is well known that the fundamental tensor g_{ij}

of a Riemannian space V_2 is a multiple of the Ricci tensor R_{ij}, provided that the curvature K of the space does not vanish identically. In fact,

$$g_{ij} = -(1/K)R_{ij}, \qquad K \neq 0. \tag{8}$$

Hence, a space V_2 with the symmetric connection Γ^i_{jk} is a Riemannian space of non-zero curvature if and only if $B_{ij} = B_{ji}$, $|B_{ij}| \neq 0$, and equations (3) have a solution of the form $g_{ij} = \rho B_{ij}$, $\rho \neq 0$. By Lemma 2, the last of the conditions is equivalent to

$$B_{ij,k} = \phi_k B_{ij}, \qquad B_{ih}B^h_{jkl} + B_{hj}B^h_{ikl} = 0.$$

But the equations of the second of these sets are identically satisfied, by virtue of the relations (2) and the fact that the Weyl tensor W^i_{jkl} vanishes identically when $n = 2$.

THEOREM 2. *A necessary and sufficient condition that a space V_2 with symmetric connection Γ^i_{jk} be a Riemannian space of non-zero curvature is that*

$$B_{ij,k} = \phi_k B_{ij}, \qquad B_{ij} = B_{ji}, \qquad |B_{ij}| \neq 0.$$

The fundamental tensor g_{ij} can then be found by quadratures and is uniquely determined to within a constant factor.

By Lemma 2 and equations (8), we have

$$g_{ij} = ce^{-\phi}B_{ij}, \qquad K = -(1/c)e^{\phi}, \qquad c \neq 0.$$

Hence, $B_{ij,k} = 0$ if and only if $\phi_k = 0$ or $K = \text{const.}$

COROLLARY. *A necessary and sufficient condition that the given space be a Riemannian space of constant curvature, not zero, is that*

$$B_{ij,k} = 0, \qquad B_{ij} = B_{ji}, \qquad |B_{ij}| \neq 0.$$

5. *Einstein Spaces.* A Riemannian space V_n for which the Ricci tensor R_{ij} is a multiple ρ of the fundamental tensor g_{ij} is known as an Einstein space. The multiplier ρ is necessarily equal to R/n, where $R = g^{ij}R_{ij}$ is the scalar curvature of the space. Thus, an Einstein space of non-zero scalar curvature is completely characterized by the relations

$$g_{ij} = (n/R)R_{ij}, \qquad R \neq 0. \tag{9}$$

By an argument similar to that employed in the case $n = 2$, we arrive at the following result.

THEOREM 3. *A necessary and sufficient condition that a space V_n with the symmetric connection Γ^i_{jk} be an Einstein space of non-zero scalar curvature is that $B_{ij} = B_{ji}$, $|B_{ij}| \neq 0$, and*

$$B_{ij,k} = \phi_k B_{ij}, \qquad B_{ih}W^h_{jkl} + B_{hj}W^h_{ikl} = 0.$$

The fundamental tensor g_{ij} can then be found by quadratures and is uniquely determined to within a constant factor.

We have, in fact,

$$g_{ij} = ce^{-\phi}B_{ij}, \qquad R = (n/c)e^{\phi}, \qquad c \neq 0.$$

When $B_{ij,k} = \phi_k B_{ij}$ is replaced by $B_{ij,k} = 0$, the conditions become those under which the given space is an Einstein space of constant scalar curvature, not zero. The corollary thus obtained is, however, of importance only in the case $n = 2$, since every Einstein space for $n > 2$ is of constant scalar curvature.

6. *Riemannian Spaces of Constant Curvature.* When $n > 2$, the Riemannian curvature of a Riemannian space V_n is constant if and only if the Weyl tensor, formed for the moment in terms of $R^i{}_{jkl}$ and R_{ij}, vanishes identically. But when $n = 2$, $W^i{}_{jkl}$ is always identically zero. Consequently, a necessary and sufficient condition that the Riemannian curvature of a Riemannian space V_n be a point function (and hence, when $n > 2$, a constant) is that $W^i{}_{jkl} = 0$.

The Riemannian spaces V_2 and the Riemannian spaces V_n, $n > 2$, of constant Riemannian curvature are Einstein spaces. We have, in fact, in both cases,

$$g_{ij} = -[1/(n-1)K]\, R_{ij}, \tag{10}$$

provided the Riemannian curvature K is not zero.

In light of these considerations, we conclude immediately, from Theorem 3:

THEOREM 4. *A necessary and sufficient condition that a space V_n with the symmetric connection $\Gamma^i{}_{jk}$ be a Riemannian space whose Riemannian curvature is a point function, not zero, is that $B_{ij} = B_{ji}$, $|B_{ij}| \neq 0$, and*

$$B_{ij,k} = \phi_k B_{ij}, \qquad W^i{}_{jkl} = 0.$$

In this case, we have

$$g_{ij} = ce^{-\phi}B_{ij}, \qquad K = -[1/(n-1)c]\, e^{\phi}, \qquad c \neq 0.$$

Here, too, there is a corollary involving the replacement of the condition $B_{ij,k} = \phi_k B_{ij}$ by $B_{ij,k} = 0$, but, as in the previous case, the corollary has significance only when $n = 2$.

It is to be noted that Theorem 4 is a special case of Theorem 3 and Theorem 2 in turn a special case of Theorem 4.*

7. *Projectively Flat Spaces.* The space V_n with the symmetric connection Γ^i_{jk} is called (absolutely) flat if its curvature tensor B^i_{jkl} vanishes, i. e. if it is identical with a Riemannian space of zero Riemannian curvature. For reasons which we need not go into here, the space V_n is said to be projectively flat if it can be put into one-to-one point correspondence with a flat space so that to the paths of the one space correspond the paths of the other.

The condition $W^i_{jkl} = 0$ of Theorem 4 is necessary and sufficient that the given space V_n be projectively flat when $n > 2$, and, when, $n = 2$, it imposes no restriction.

Furthermore, if a projectively flat space V_n is Riemannian, it can be put into one-to-one point correspondence with a Riemannian space of constant (zero) Riemannian curvature so that geodesics correspond, and hence is itself of constant Riemannian curvature, by the theorem of Beltrami.

THEOREM 5. *A necessary and sufficient condition that a projectively flat space V_n with the symmetric connection Γ^i_{jk} be a Riemannian space of non-zero Riemannian curvature is that*

$$B_{ij,k} = \phi_k B_{ij}, \qquad B_{ij} = B_{ji}, \qquad |B_{ij}| \neq 0.$$

If the conditions $B_{ij,k} = \phi_k B_{ij}$ are replaced by $B_{ij,k} = 0$, the theorem remains valid, for in this case the Riemannian space must be of constant Riemannian curvature for $n = 2$ as well as for $n > 2$. The alternative conditions are, however, more restrictive than the original ones.†

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* The author is indebted to J. M. Thomas for pointing out that his original method of treatment, which was confined to the case $n = 2$, could be applied equally well, when $n > 2$, to projectively flat spaces. The present method of treatment, based on the general Theorem 1 and the relations (2), resulted from generalizing the original method so that it would apply also to the case in which (1) is assumed to be the linear element of an Einstein space.

† The alternative theorem is known; cf. Eisenhart, *loc. cit.*

The Regular Components of Surface Transformations.

By Paul A. Smith.

On a closed surface which is subjected to a one-one continuous transformation into itself, there exists a certain closed invariant point set towards which the remaining points of the surface are carried when the transformation is indefinitely iterated. In a variety of cases, it is possible to determine on the surface certain open sets characterized by a property of "uniform motion" towards the invariant central set. These so-called regular components were first considered and their structure studied by Birkhoff and Smith,* but only for the case in which the surface was topologically equivalent to a sphere; in this case the treatment is relatively simple since it is possible to avoid extensive reference to the facts and methods of surface topology. In the present study of regular components, we shall make virtually no restriction on the surface undergoing transformation save that it be closed. Regular components will be classified with regard to their connectivity and their motion under iteration. It will appear that the number of possible types of components is independent of the connectivity of the surface on which they exist. With one exception, each type of component found on a given surface may equally well exist on every other surface. Transformations which admit regular components of the exceptional type are of very special structure; they are analyzed in some detail in the final section.

1. *Notation and definitions.* We shall consistently denote by S a closed surface of arbitrary connectivity and orientability, and by T an arbitrary $(1,1)$ continuous transformation of S into itself.

In introducing certain definitions we shall require the use of a metric on S. For definiteness we shall suppose S to be a suitably regular surface situated in a Euclidean space R_n and to derive its metric from that of R_n. The distance between the two points of S will be taken to mean the geodesic distance on S; there will be no ambiguity since only arbitrarily small distances will be considered.

Such expressions as "circuit on S" or "region on S" will have a purely point-set theoretic significance. For example, a circuit on S will mean a

* "Structure Analysis of Surface Transformations" (to be denoted in this paper by "S. A."), *Journal de Mathématiques*, Vol. 7 (1928), pp. 345-379.

subset of S which constitutes a simple closed curve in R_n, and a region on S will mean a connected subset of S, the points of which are inner points relative to S.

A region on S which is homeomorphic to a region in a Euclidean plane, is a region of *planar type*. In particular, if a region R is homeomorphic to the plane region interior to a circle, R is simply connected, and if R is homeomorphic to the region between two concentric circles, R is *doubly connected*. A doubly connected region on S whose boundary consists of two non-intersecting circuits will be called a *ring*.

In recalling the following facts concerning the connectivity of regions on S, we are drawing freely on well established theorems of surface topology.*

A necessary and sufficient condition that a region R on S be of planar type is that R shall be a proper subset of S, and that an arbitrary circuit α in R shall separate it into two regions; and a necessary and sufficient condition that R be simply connected is that for every choice of α, one of the two regions shall be a 2-cell.

If R is doubly connected, there can of course be drawn in R a circuit which is not the boundary of a 2-cell contained in R. If R is of planar type, but not simply or doubly connected (*i. e.*, is of "higher order of connectivity"), there can be drawn in R a pair of non-intersecting circuits neither of which bounds a 2-cell in R, and which together fail to bound a ring in R.

2. We shall recall a few additional facts of surface topology preliminary to proving a useful lemma.

Let U be a surface either closed or with boundary.† A fundamental topological invariant of U is its characteristic $C(U) = \alpha_0 - \alpha_1 + \alpha_2$ where α_0, α_1, and α_2 are respectively the numbers of 0—, 1—, and 2-cells which form an arbitrary subdivision of U. If U possesses a boundary, then $C(U) \leq 1$. In particular if $C(U) = 1$, then U (minus its boundary) is equivalent either to a 2-cell or to a Möbius strip; and if $C(U) = 0$, U is a ring (§ 1). For all other types of surfaces with boundary, $C(U) < 0$.

Suppose now that S be separated by means of a finite number of non-intersecting 2-sided circuits into two or more regions $V^1, \cdots, V^n$. If we call U^i the surface with boundary consisting of V^i plus its boundary points, it is readily verified that

$$C(S) = C(U^1) + \cdots + C(U^n).$$

* See, for example, Kerékjártó, "Vorlesungen über Topologie I."

† A surface with boundary is homeomorphic to a closed surface from which there have been removed a finite number of 2-cells the boundaries of which are mutually exclusive.

Hence if k of the regions V^i are Möbius strips, the contribution of the corresponding U's toward $C(S)$ is k. Let us assume that none of the remaining $n-k$ regions is a 2-cell or a ring. Then the contribution of the corresponding U's is $\leq (n-k)$ and we have $C(S) \leq k-(n-k)$ and $n \leq 2k - C(S)$. Now the largest possible value k' of k relative to all possible subdivisions of the type considered depends only on the connectivity index of S. Hence, under the assumption that no region of the subdivision is a ring or a 2-cell, we have that $n \leq \eta(S) = 2k' - C(S)$; $\eta(S)$ depends only on the topological invariants of S. Consequently, if a given S be separated into more than $\eta(S)$ regions, one at least is a 2-cell or a ring.

From this we readily prove the following

LEMMA. *For a given S, there can be determined an integer $N > 1$ such that of every set of N or more mutually exclusive 2-sided non-cell-bounding circuits on S (if any such exist), at least one pair bounds a ring on S.*

Proof. It is a simple exercise in surface topology to show that given an integer $K > 1$, there can be chosen an integer L such that every set of L or more mutually exclusive circuits of the stated type on S separate S into at least K regions. If in particular $K = \eta(S) + 1$, one of the regions is a ring and the corresponding value for L may therefore be taken for N.

It is clear that the smallest possible value N^* of N for a given S is the same for every surface homeomorphic to S, and depends therefore only on the invariants of S. The explicit form of this dependence can easily be derived. It will be sufficient for our purposes, however, to point out that for a torus, $N^* = 2$; in fact, a torus is separated by every pair of non-intersecting non-cell-bounding circuits into two rings.

3. *Regular points.* We shall now state briefly the essential definitions and theorems relative to the motion of points under indefinite iteration. For more completeness of detail, the paper referred to above (S. A.) may be consulted.

Let $T_2, T_3, \cdots$ be the successive powers of $T(=T_1)$ and $T_{-2}, T_{-3}, \cdots$ those of the inverse T_{-1}. If E is an arbitrary point set on S, we shall denote by E_n $(n = \pm 1, \pm 2, \cdots)$ the transformed set $T_n(E)$.

Suppose that a given region σ on S has the property that it is intersected by none of its images $\cdots \sigma_{-2}, \sigma_{-1}, \sigma_1, \sigma_2, \cdots$. Then no two images of σ can intersect. For if σ_p and σ_q $(p \neq q)$ have points in common, so do σ and σ_{p-q}. Any region of type σ on S is called a *wandering region*, and any point contained in a wandering region is a *wandering point*; the totality of

wandering points constitutes an open set W. The closed set $M = S - W$ is non-null and consists of the *non-wandering* points of S; M is invariant under T,—that is, M and $T(M)$ are identical point sets (S. A., p. 351).

An important property of wandering points is the following: if P is an arbitrary wandering point and (P_n, M) is the distance from P_n to M, then $\lim_{n\to\pm\infty} (P_n, M) = 0$ (S. A., p. 351), or, as we shall say, P converges toward M under indefinite iteration of T as well as T_{-1}.

We shall be interested in the motion of sets of wandering points, especially as regards the uniformity of their convergence toward M. Suppose that a region σ of wandering points has the property that there exists a positive integer h such that the set $\sigma_h + \sigma_{h+1} + \cdots$ (or $\sigma_{-h} + \sigma_{-h-1} + \cdots$) is entirely contained in an arbitrarily small neighborhood V of M, h depending only on the choice of V. We shall call a point contained in such a region an ω- (*or* α-) *regular point;* wandering points contained in no such region are ω- (α-) *irregular.*

Suppose that a closed set E consist entirely of ω- (α-) regular points. It follows from an application of the Heine-Borel theorem that for $n \geqq N$ each point of $E_n (E_{-n})$ is contained in an arbitrarily small neighborhood V of M, where N depends only on the choice of V.

The totality of ω- (α-) regular points, if any exist, constitutes an open set, any maximal connected subset of which we shall call an ω- (α-) *regular component.** The image under T or T_{-1} of an ω- (α-) regular component is again an ω- (α-) regular component and from this it follows readily that a regular component C is either a wandering region, or else there exists an integer $k > 0$ such that C_p and C_q are identical if $p \equiv q \pmod{k}$ and mutually exclusive if $p \not\equiv q \pmod{k}$. We shall call regular components of the latter type *periodic,* of order k. It is our purpose to study the structure of components of both types.

THEOREM 1. *If the boundary ρ of a region R on S consists of ω-regular points, and if for some integer $h > 0$ the regions R, R_h, R_{2h}, $\cdots$ are mutually exclusive, then each point of R is ω-regular. A similar theorem holds for the α-regular case.*

Proof. The regions in each of the h sequences

$$R_i,\ R_{i+h},\ R_{i+2h}, \quad (i = 0, 1, \cdots, h-1)$$

* We shall frequently use the term "regular" with reference to a point or component which is ω- or α-regular or both. In all proofs, however, "regular" will mean ω-regular for definiteness.

are mutually exclusive. Since the surface measure mS of S is finite (see § 1), it follows that as $n \to \infty$, $\lim m R_{i+nh} = 0$ for $i = 0, 1, \cdots h-1$, and hence $\lim mR_n = 0$. Therefore, if ϵ is an arbitrary positive number, each point of R_n is within a distance $< \epsilon$ from ρ_n whenever n is greater than a suitably chosen positive integer N. Since the points of the closed set ρ are ω-regular, each point of ρ_n is within the distance ϵ from M, whenever n is greater than K, suitably chosen. Each point of R_n is therefore within a distance of 2ϵ from M whenever $n > N + K$. It follows that all wandering points contained in R are ω-regular.

The proof of the theorem will be complete if we show that each point of R is wandering. Suppose on the contrary that R contains points of M. Then (S. A., theorem 4, p. 360.) R is intersected by infinitely many of its images $R_1, R_2, \cdots$. Let one of these intersecting images be R_p, where p is chosen greater than $N + K$ so that each point of R_p is within a distance 2ϵ from M. Now ρ is at a non-zero distance δ from M, and ϵ being arbitrary, we may assume that $2\epsilon < \delta$. Hence R_p fails to meet ρ and is therefore entirely contained in R. We have then that $R \supseteq R_p \supseteq R_{2p} \supseteq \cdots \supseteq R_{hp}$. But this is impossible, since R and R_{hp} are mutually exclusive. This completes the proof.

THEOREM 2. *If a component D (i. e., a maximal connected subset) of $S - M$ is a wandering region, then D is an ω- and α-regular component.*

The proof is similar to that of the preceding theorem. We have $\lim_{n \to \pm\infty} mD_n = 0$. Hence, since the boundary of each D_n is in M, the totality of points in any region contained in D is brought within and remains within an arbitrarily small distance of M on indefinite iteration of T or T_{-1}.

4. *Generating Rings.* Suppose there exists on S a ring r whose boundary circuits α and β are transforms one of the other under a power of T,—say $\beta = \alpha_m$ $(m > 0)$. Suppose further that r is such that the rings $r, r_m, r_{2m}, \cdots$ are mutually exclusive. We shall call r a *generating ring* of order m. It is clear that the rings of the sequence

$$\cdots, r_{-m}, r, r_m, \cdots$$

are mutually exclusive and any two successive rings are adjacent in the sense that their boundaries have a circuit in common. Hence the set

$$R = r' + r'_m + \cdots + r'_{-m} + \cdots \qquad (r' = r + \alpha + \beta)$$

is a doubly connected region invariant under T_m, and may be considered as being generated by r.

THEOREM 3. *Neither circuit of the boundary of a generating ring can be entirely contained in a regular component C of wandering type.*

Suppose on the contrary that $\alpha + \alpha_m$ is the boundary of a generating ring r and that α is contained in C. Then since the points of α and α_m are regular, so are those of r by theorem 1, and r therefore contains no boundary points of C or C_m. Hence r is contained in C since α is, and also in C_m since α_m is. But this is impossible, because C and C_m are mutually exclusive.

THEOREM 4. *If S is not a torus and if there is on S a sequence of mutually exclusive circuits of the type* $\alpha, \alpha_k, \alpha_{2k}, \cdots$ *some pair of which bounds a ring on S, then there exists a generating ring on S, one of whose boundary circuits is* α.

The theorem holds also if S is a torus, provided α *is the boundary of a 2-cell on S.*

Proof. Suppose that α_{pk} and α_{qk} $(p < q)$ bound a ring. Then α and α_m, $m = (q - p)k$, bound a ring r.

There are several cases to be considered. In the first place, if the rings $r, r_m, r_{2m}, \cdots$ are mutually exclusive, r is the desired generating ring.

Next, suppose that r and r_m are mutually exclusive while r_{am} and r_{bm} $(a < b)$ overlap. Let $h > 1$ be the smallest integer such that r and r_{hm} overlap. The mutually exclusive rings $r, r_m, \cdots r_{(h-1)m}$ together with their boundaries may be considered as constituting a tube the two ends of which overlap on the addition of r_{hm}. Thus a torus is formed which must be identical with S and on which α is non-cell-bounding. This situation is contrary to the hypothesis and the case under consideration is therefore not possible.

Suppose finally that r and r_m overlap. Since α, α_m, and α_{2m} are mutually exclusive, this can happen in only two ways:

(a) One of the two rings is contained in the other.

(b) The two rings have in common a ring s, a proper part of both and bounded by α_m and α_{2m} (i. e., the outer edges of the rings overlap).

Case (b) however is not possible, for the rings s, $r - s$, and $r_m - s$ taken together constitute a torus which must be identical with S and on which α is non-cell-bounding, contrary to hypothesis.

This leaves only case (a). Assume, say, that r contains r_m. Then also

$$r_m \supset r_{2m} \supset \cdots. \tag{1}$$

Let ρ be the ring $r - (r_m + \alpha_{2m})$. From the relation $\rho \subset r$, we have

$$\rho \subset r, \qquad \rho_m \subset r_m, \cdots. \tag{2}$$

From (1) and (2), each ring of the sequence $\rho_m, \rho_{2m}, \cdots$ is contained in r_m and therefore fails to intersect ρ. Hence the rings $\rho, \rho_{2m}, \rho_{4m}, \cdots$ are mutually exclusive, for if ρ_{2am} and ρ_{2bm} intersect, so do ρ and $\rho_{(2b-2a)m}$. Therefore, since the boundary of ρ is $\alpha + \alpha_{2m}$, ρ is the desired generating ring. This completes the proof of the theorem.

5. *The connectivity of the regular components.*

THEOREM 5. *A regular component is a region of planar type.*

Let C be a regular component and α an arbitrary circuit in C. We must show (§ 1) that C is separated by α.

Let us assume the contrary and suppose first that α is two-sided. If τ is a small arc crossing α at Q, the two end-points of τ can be joined by a second arc contained in C and intersecting neither α nor τ. Thus we obtain a circuit β contained in C, crossing α at Q, and having no other point in common with α. Clearly, both α and β are non-cell-bounding on S.

Since the points of α are ω-regular, the successive images $\alpha_1, \alpha_2, \cdots$ converge uniformly (§ 3) toward M. Since the closed set $\alpha + \beta$ is at a non-zero distance from M, a positive integer K can be chosen such that α_n fails to meet $\alpha + \beta$ for each $n \geq K$. The circuits $\alpha_k, \alpha_{2k}, \cdots$ are mutually exclusive. For if α_{pK} and α_{qK} $(p < q)$ intersect, so do α and $\alpha_{(q-p)K}$ which is impossible by the choice of K. Moreover, these circuits are all non-cell bounding and two-sided, since α is. Hence by the lemma (§ 2) two circuits, say α_{mK} and α_{nK} $(m < n)$ can be chosen which bound a ring on S. Hence α and $\alpha_{(n-m)K}$ bound a ring, say R. Now if we trace a complete circuit on β in the proper sense, we shall enter R by crossing α at Q; but on leaving R, we cannot again cross α and therefore β crosses $\alpha_{(m-n)K}$ which is impossible. This disposes of the case where α is assumed to be two-sided.

If α is one-sided, so are the circuits $\alpha_1, \alpha_2, \cdots$. However, the reasoning above which led to the integer K still applies and we have therefore an infinite sequence $\alpha, \alpha_K, \alpha_{2K}, \cdots$ of mutually exclusive one-sided circuits. But this is impossible, since there can exist on S at most N mutually exclusive one-sided circuits, where N is finite and depends only on the invariants of S. This completes the proof.

We shall say that a regular component of C is *simple*, if every circuit in C is the boundary of a 2-cell on S.

THEOREM 6. *A simple regular component of wandering type is simply connected.*

Let C be such a component, and Γ its boundary. Also let α be an

arbitrary circuit in C and A a 2-cell with boundary α. To demonstrate the theorem it will be sufficient to show that A is contained in C (§ 1).

Suppose this were not the case. Then A contains points of Γ and each of these is either irregular or non-wandering. It follows then from theorem 1 that A cannot be wandering. Let h be a positive integer such that A and A_h intersect. Since α and α_h fail to intersect, one of the following situations holds:

(a) One of the cells A, A_h together with its boundary is contained in the other.

(b) The part common to A and A_h is a ring bounded by $\alpha + \alpha_h$.

We shall first consider the case (a). Suppose for example that $A \supset A_h + \alpha_h$. Then we have $A \supset A_h \supset A_{2h} \supset \cdots$, and if we call r the ring $A - (A_h + \alpha_h)$, the images r, r_h, r_{2h}, $\cdots$ are mutually exclusive. The boundary of r is $\alpha + \alpha_h$ and hence r is a generating ring, one of whose boundary circuits, namely α, is in C. This is impossible by theorem 3.

(b) In this case, A and A_h together cover S, and S is equivalent to a sphere. The cells A_h and A_{2h} overlap just as A and A_h do, and it is readily seen that the cells A and A_{2h} therefore overlap in such a way that one is contained in the other. Thus we are reduced to the case (a) with h replaced by $2h$. This completes the proof.

THEOREM 7. *If S is not a torus, every regular component of wandering type on S is simple.*

For, let C be such a component. If C is not simple, a circuit α can be chosen in C which fails to bound a cell on S. The mutually exclusive non-cell-bounding circuits α, α_1, α_2, $\cdots$ are two-sided since C is of planar type (theorem 5). Hence by the lemma (§ 2), there exist integers a and b $(0 < a < b)$ such that α_a and α_b bound a ring on S. Hence by theorem 4 there exists a generating ring one of whose boundary circuits is α. But since α is in C, we have a contradiction to theorem 3.

THEOREM 8. *On a surface which is not a torus, every regular component of wandering type is simply connected.*

This is a consequence of theorems 6 and 7.

THEOREM 9. *A regular component which contains at least one boundary circuit of a generating ring r is doubly connected.*

We shall show in fact that C is identical with the doubly connected limit region

$$R = r' + r'_h + \cdots + r'_{-h} + \cdots$$

where h is the order of r (§ 4).

Let the boundary of r be $\alpha + \alpha_h$ where one of these circuits, which we may assume to be α, is contained in C. Also let the boundary of R be $\rho^1 + \rho^2$ where ρ^1 and ρ^2 are respectively the continua toward which the sequences r_h, $r_{2h}, \cdots$ and r_{-h}, $r_{-2h}, \cdots$ converge.

Assuming definitely that C is ω-regular, we have immediately that $\rho^1 \subseteq M$. For, an arbitrary point Q is a limit point of a sequence Q^h, $Q^{2h}, \cdots$ of points suitably chosen on the respective circuits α_h, $\alpha_{2h}, \cdots$. These circuits however converge uniformly toward M (§ 3) and it follows that Q is a point of M.

Next, the points of ρ^2 which are not contained in M, if any exist, are ω-irregular. For, let P be such a point and σ a neighborhood of P chosen so small that it contains no points of M. Now σR converges toward ρ^1 in iteration of T_h, whereas the points P, P_h, $P_{2h}, \cdots$ are all on ρ^2. Hence the successive images σ, $\sigma_h, \cdots$ tend to stretch across the region R, and must therefore, from a certain rank on, all intersect the circuit α which is at a non-zero distance from M. Hence σ does not converge uniformly toward M on iteration of T and P is therefore ω-irregular.

It follows that although C contains points of R, for example α, C contains no boundary points of R. Hence $C \subseteq R$.

We have finally to show that $R \subseteq C$. It follows from theorem 1 that the points of r' and hence of every image of r' are ω-regular. Thus R consists only of ω-regular points and since some points of R are in C (e. g. α), R is entirely contained in C. This completes the proof.

THEOREM 10. *Every regular component is at most doubly connected.*

Let C be a regular component and α an arbitrary circuit in C; by theorem 5, α is two-sided. If α is the boundary of a 2-cell contained in C, then C is simply connected (§ 1); in the contrary case C is doubly connected as we proceed to prove.

Suppose that α is not the boundary of a cell contained in C. Since α consists of ω-regular points and is at a non-zero distance from M, a positive integer N can be chosen (§ 3) such that each α_{N+n} $(n = 0, 1, \cdots)$ fails to meet α. Hence the circuits α, α_N, $\alpha_{2N}, \cdots$ are mutually exclusive by the familiar argument.

We shall consider separately the cases in which α does or does not bound a cell on S.

Suppose first that α is the boundary of a 2-cell A. Since A is not

contained in C, it contains points which are not ω-regular, namely, boundary points of C. Hence by theorem 1, the cells A, A_N, $A_{2N}, \cdots$ cannot be mutually exclusive. Suppose then that A_{aN} and A_{bN} intersect. Then one of these cells together with its boundary is contained in the other, or else the part common to A_{aN} and A_{bN} is a ring. In either case $\alpha_{aN} + \alpha_{bN}$ is the boundary of a ring on S. Since α is cell-bounding, theorem 4 is applicable even when S is a torus with the result that there exists a generating ring on S one of whose boundary circuits is α. Hence by theorem 9, C is doubly connected.

There remains the case in which α is non-cell-bounding. Then α_N, $\alpha_{2N}, \cdots$ are non-cell-bounding and also two-sided. Hence by the lemma (§ 2) a pair of these circuits can be chosen which bounds a ring on S. If S is not a torus, theorem 4 is again applicable and then, as above, theorem 9, with the result that C is doubly connected.

In the excepted case, α may be assumed to bound a cell on S, as we shall show, and then we are reduced to the case already treated. For assume S to be a torus. Then *if every circuit α which fails to bound a cell in C, also fails to bound a cell on S, C* is at most doubly connected. To prove this, suppose that C is neither simply nor doubly connected. Then there exists in C (§ 1) a pair of non-intersecting circuits β and γ neither of which is the boundary of a cell in C, and such that $\beta + \gamma$ is not the boundary of a ring in C. By assumption, β and γ are non-cell-bounding on S and hence (§ 2) separate S into two rings, r^1 and r^2, and both of these contain boundary points of C. Let β and γ be joined by a simple arc in C. A suitably chosen sub-arc τ will connect β and γ and will be entirely contained in r^1 or r^2, say r^1. On making a cut along τ, r^1 is reduced to a simply connected region r^* which contains boundary points of C and whose boundary is contained in C. On tracing a curve in r^* close to its boundary we obtain a circuit which fails to bound a cell in C, but bounds a cell on S. This contradiction completes the proof of the theorem.

6. *Transformations of a Torus.* We have seen that regular components of wandering type are simply connected if S is not a torus. On a torus, however, such components may actually be doubly connected. For example let S be a circular torus and θ and ϕ its angular co-ordinates. It is possible to define a transformation t of the circle $\phi = 0$ into itself which admits a regular component (arc segment) of wandering type.† If each circle ϕ

† For a detailed description of such a transformation, see Birkoff, "Quelques Théorèmes sur le Mouvement des Systèmes dynamiques," *Bulletin de la Société Mathématiques de France*, Vol. 40, p. 16.

$=$ const. be subjected to a transformation congruent to t, we have a transformation T of the desired type. The regular components are wandering rings bounded by circles of the type $\theta =$ const.

Not only is the existence of regular components of the type under consideration restricted to surfaces of special type, but their existence implies a certain speciality in the structure of T itself as we shall show. In the remaining paragraphs we shall assume that S is a torus and that T admits regular components of doubly connected wandering type. We shall denote by D an arbitrarily chosen one of these components.

I. *The closed set $E = S - D$ contains no invariant or periodic points of continua.*

Suppose on the contrary that E does contain a periodic continuum, ξ say, of order h. By theorem 6, D is non-simple and hence there exists in D a circuit α which is non-cell-bounding; α fails to meet ξ, and ξ being invariant under T_h, each image $\alpha_h, \alpha_{2h}, \cdots$ fails to meet ξ. The mutually exclusive circuits α and α_h separate S into two rings (§ 2) ρ and σ, one of which, say ρ, fails to contain points of ξ; hence each of the rings $\rho_h, \rho_{2h}, \cdots$ fails to contain points of ξ.

The rings ρ and ρ_h are mutually exclusive. For suppose they intersect; then one is contained in the other, or else the edges along α and α_{2h} respectively overlap. But this latter situation is impossible, since ρ and ρ_h taken together would cover S, whereas neither ρ nor ρ_h contains points of ξ. In the former case, we are led by the reasoning in the proof of theorem 4, case (a), to the existence of a generating ring on S, one of whose boundary circuits is α. This contradicts theorem 3.

Thus ρ and ρ_h fail to intersect. Moreover, ρ is intersected by none of its images $\rho_h, \rho_{2h}, \cdots$. For on adding successive rings to ρ, we build a tube and at no stage will the two ends overlap for otherwise a finite number of rings would cover S, whereas none of them contains points of ξ. Hence ρ is a generating ring which again contradicts theorem 3. This completes the proof.

Since D itself contains no invariant or periodic points, we have as a corollary that *there are no invariant or periodic points whatever on S.*

II. *T admits no regular components of wandering type.*

For suppose the regular component E is periodic. Since E is at most doubly connected, its boundary consists of two continua (which may intersect). These continua are contained in $S - D$ and both are invariant or periodic, which is impossible by I.

Let α continue to represent an arbitrary non-cell-bounding circuit in C and let Q^n be an arbitrarily chosen point on α_n $(n = 1, 2, \cdots)$. A limit point of the sequence $Q^1, Q^2, \cdots$ belongs necessarily to M on account of the uniform manner in which α converges toward M on indefinite iteration of T (§ 3). Let L be the totality of the limit points obtained from all possible sequences of the type $Q^1, Q^2, \cdots$; L is clearly a closed invariant subset of M. Moreover, L is independent of the choice of D and α; in fact L *is identical with* M as will presently appear.

Let E be an arbitrary component of the open set $S - L$; clearly E, like D, is either wandering or periodic.

A 2-cell B on S whose boundary β is in E cannot contain boundary points of E. For suppose B contains a boundary point Q. Let $Q^1, Q^2, \cdots$ be a point sequence of points which has Q for a limit point and which is of the type which determines L. Infinitely many points of this sequence are in B, and the corresponding circuits of the sequence $\alpha_1, \alpha_2, \cdots$, since they are non-cell-bounding, all intersect β. Hence on β there is at least one point of L, which is impossible.

By a similar argument, L *is perfect.* For suppose that Q is a point of L and B an arbitrary 2-cell containing Q. Then on the boundary of B there is at least one point of L, and hence Q is a limit point of L.

The set L is nowhere dense. For suppose that L contains an inner point P. Let T be that maximal connected subset of inner points of L which contains P. Since F contains only non-wandering points, it is intersected by some of its images, one of which, say, is F_h. But since F is maximal, we have $F \equiv F_h$. Thus F together with its boundary is a periodic continuum in $S - D$ which is impossible by I.

Since the boundary of E is in M, it follows by theorem 2, § 3, *that E is an ω- and α-regular component.*

We are now prepared to show that L is identical with M. Since $L \subseteq M$, it remains only to show that $M \subseteq L$. Suppose on the contrary that there is a point P of M which is not in L. Then P is contained in one of the components of $S - L$, say in H. But this is impossible, for by the preceding paragraph, H contains no non-wandering points.

Finally, we shall show that *each regular component is identical with a component of $S - L$.*

Let C be an arbitrary regular component. Then there exists a component F of $S - M$ such that $C \subseteq F$. But since $M \equiv L$, F is also a component of $S - L$ and therefore is a regular component, as we have shown. It follows that C cannot be a proper part of F and hence $C \equiv F$.

The preceding result is interesting for the following reason: since C is identical with a component of $S - L$, the boundary of C consists only of non-wandering points. Hence *T admits no irregular points.*

We now summarize the results of this section in the following

THEOREM. *If T admits a regular component of doubly connected wandering type, then* (a) *S is a torus;* (b) *M is a perfect nowhere dense set which contains no invariant or periodic continua;* (c) *T admits no invariant or periodic points;* (d) *T admits no irregular points;* (e) *the regular components of T are both ω- and α-regular and there are none of periodic type.*

A surface transformation T with these properties is in a sense analogous to a transformation of a simple closed curve with rotation number * incommensurable with 2π and non-wandering points constituting a perfect nowhere dense set. It is possible in fact to attach a unique rotation number to T itself, as we shall indicate. Let α be an arbitrary non-cell-bounding circuit in C. Any two circuits of the sequence

$$\cdots, \alpha_{-1}, \alpha, \alpha_1, \cdots$$

separate S into two rings and these circuits can therefore be ordered cyclically on S. Now let $\delta^1, \delta^2, \cdots$ be an infinite sequence of mutually exclusive arcs on a circle σ, so chosen that the set $\sigma - \Sigma\delta^i$ is perfect and nowhere dense. By establishing a $(1,1)$ correspondence H between the circuits α_i and the arcs δ^j in such a way as to preserve order, a transformation of the set $\Sigma\delta^i$ into itself is induced by T, and this transformation extends by continuity to the whole of σ; its rotation number λ is incommensurable with 2π. Moreover λ is independent of the choice of the correspondence H and is therefore characteristic of T. We shall not, however, carry this discussion into further detail.

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* Rotation numbers were first introduced by Poincaré; for a definition see Birkhoff, "Surface Transformations and Their Dynamical Applications," *Acta Mathematica*, Vol. 43, p. 87.

Polygenic Functions of the Dual Variable $w = u + jv$.

By Edna E. Kramer.

It is the purpose of this paper to develop a theory *analogous*, and, roughly speaking, *dual* to Kasner's recent theory of polygenic functions of z.* There seems to be no general principle of transference whereby the properties of the functions here considered could be directly inferred from the corresponding facts in the original theory. A summary of some of the results obtained below will serve to illustrate points of similarity and difference in the two theories.

The first derivative of a polygenic function of z is represented by a congruence of *circles*. For functions of the dual variable we have a congruence of directed circles, or *cycles*.† In the matter of the method of generation of these curves, the theories diverge. Kasner has found it appropriate, with this question in view, to speak of a congruence of *clocks*. It will be shown below that our derivative cycles are described in an entirely different way. There is also some variation in the nature of the specialization of the congruences. For the second derivative we have as corresponding pictures, a *line* and a *cycle*, a curve of *eighth* order and a directed curve of *sixth* order.

1. *The First Derivative.* Just as the *conformal* group is represented by the monogenic functions of the ordinary complex variable $z = x + iy$, the *equilong* group ‡ is represented by the monogenic functions of the dual variable

$$w = u + jv, \text{ where } j^2 = 0,$$
$$u = tg(\theta/2), \qquad v = (\rho/2)\sec^2(\theta/2),$$

* E. Kasner, "Theory of Polygenic Functions," *Science*, Vol. 66 (1927), pp. 581-582, and *Proceedings of the National Academy of Sciences*, Vol. 14 (1928), pp. 75-82; "Second Derivative of a Polygenic Function," *Transactions of the American Mathematical Society*, October 1928; "Note on the Derivative Circular Congruence of a Polygenic Function," *Bulletin of the American Mathematical Society*, Vol. 34 (1928), pp. 561-565; also L. Hofmann and E. Kasner, "Homographic Circles or Clocks, with an Appendix on Polygenic Functions," by E. Kasner, the same *Bulletin*, Vol. 34, pp. 495-503.

† We shall deal, in general, with *directed* curves, and refer to a directed circle as a *cycle*, and to a directed line as a *ray*.

‡ G. Scheffers, "Isogonalkurven, Aquitangentialkurven, und Komplexe Zahlen," *Mathematische Annalen*, 1905.

and θ and ρ are oriented Hessian line co-ordinates. (If t is the perpendicular from the origin O upon a given ray s, oriented relative to s as the positive X to the positive Y axis, then ρ is equal to the distance on t from O to the intersection of s and t, and is positive or negative according as s indicates positive or negative rotation about O, and θ is the angle made by t with a fixed ray through O.)

$W = \phi(u, v) + j\psi(u, v)$ is a *monogenic* function of w, or represents an equilong transformation if

$$\phi_v = 0, \qquad \phi_u = \psi_v, \tag{1}$$

which are the analogue of the Cauchy-Riemann equations for functions of z.

We give a definition of *polygenic* functions of w analogous to that which Kasner has given for polygenic functions of z.

We shall call those functions $W = \phi(u, v) + j\psi(u, v)$ polygenic, in which ϕ and ψ are continuous, and have continuous partial derivatives in the region considered, but do not necessarily satisfy the equations (1).

The limit of the ratio $\Delta W/\Delta w$ depends, in general, not only on the ray $u + jv$, but also on the point in the ray through which the neighboring ray passes as it approaches coincidence with the given ray. Thus the derivative dW/dw has, in general, infinitely many values for a given ray, unless equations (1) are satisfied, when there is a unique value.

Now

$$\begin{aligned} dW/dw &= [\phi_u + j\psi_u + v'(\phi_v + j\psi_v)]/(1 + jv') \\ &= \phi_u + v'\phi_v + j[\psi_u + v'(\psi_v - \phi_u) - v'^2\phi_v]. \end{aligned}$$

Let $\alpha + j\beta = dW/dw$. Then

$$\alpha = \phi_u + v'\phi_v, \qquad \beta = \psi_u + v'(\psi_v - \phi_u) - v'^2\phi_v'. \tag{2}$$

Eliminating v' between these two equations, we have

$$\beta = (\phi_v\psi_u - \phi_u\psi_v)/\phi_v + [(\psi_v + \phi_u)/\phi_v]\alpha - (1/\phi_v)\alpha^2, \tag{3}$$

the equation of a *cycle*.* If (A, B) is the center, and C the radius

$$\begin{aligned} A &= (\phi_v\psi_u - \phi_u\psi_v + 1)/\phi_v, \\ B &= (\phi_u + \psi_v)/\phi_v, \\ C &= (\phi_v\psi_u - \phi_u\psi_v - 1)/\phi_v. \end{aligned} \tag{4}$$

To each ray of the plane corresponds, in general, a cycle. To the ∞^2 rays correspond, in general, ∞^2 cycles.

The above work breaks down only if $\phi_v = 0$.

* The equation of a cycle in the coordinate system used here is $v = a + bu + cu^2$.

If $\phi_v = 0$ and $\phi_u = \psi_v$, that is, if the function is monogenic, we have the ray

$$\alpha = \phi_u, \qquad \beta = \psi_u.$$

Otherwise we have the two points at infinity whose equation is

$$\phi_u\psi_v - (\psi_v + \phi_u)\alpha + \alpha^2 = 0.$$

If $\phi_u = -\psi_v$, that is, if the transformation represented by W is indirect equilong, these are the points at infinity $\alpha = \pm \phi_u$.

These conditions may be satisfied for special rays only. For such rays, then, the derivative cycle will degenerate into a ray or a point at infinity. If the conditions are identically satisfied, this is true for all rays.

The following theorems deal with special cases:

I. The congruence of cycles will become a congruence of points if

$$\phi_u\psi_v - \phi_v\psi_u = -1;$$

that is, if the Jacobian of ϕ and ψ is equal to -1.

II. The centers of the cycles will lie on the y-axis if the Jacobian of ϕ and ψ is equal to 1.

III. The centers of the cycles will lie on the x-axis if $\phi_u = -\psi_v$, so that $\phi = \chi_v$, $\psi = -\chi_u$; that is, $W = \chi_v - j\chi_u$, where χ is an arbitrary function of u and v.

IV. The ∞^2 cycles will reduce to ∞^1 with a common center at the origin if both the conditions given in II and III are satisfied.

V. If we require that the congruence be such as to include the reverse cycle to each one of the set (and hence to become a congruence of circles) we find that this cannot occur unless the congruence is one of points, which is the condition in I.

VI. Two polygenic functions $\Phi + j\Psi$ and $\phi + j\psi$ are related as follows, if for each ray of the plane the derivative cycles for the two functions are opposite cycles of the same circle:

$$(\Phi_u + \Psi_v)/\Phi_v = (\phi_u + \psi_v)/\phi_v,$$
$$(\Phi_v\Psi_u - \Phi_u\Psi_v)/\Phi_v = 1/\phi_v,$$
$$1/\Phi_v = (\phi_v\psi_u - \phi_u\psi_v)/\phi_v.$$

The last two equations are equivalent to the condition that the Jacobians be reciprocal.

VII. If the cycles are to be tangent to $w = 0$, the Jacobian of ϕ and ψ must vanish.

VIII. If the cycles are to be tangent to any fixed ray (a, b) the function must satisfy

$$\phi_v\psi_u - \phi_u\psi_v + a(\psi_v + \phi_u) - a^2 = b\phi_v.$$

IX. If the ∞^2 cycles are to reduce to the ∞^1 cycles tangent to $w = 0$ at the origin, the Jacobian must vanish, and in addition

$$\phi_u = -\psi_v.$$

X. The condition that all the centers lie at a fixed point is

$$\phi_v\psi_u - \phi_u\psi_v + 1 = a\phi_v, \qquad \psi_v + \phi_u = b\phi_v.$$

XI. The congruence will reduce to a single cycle if

$$\phi_v\psi_u - \phi_u\psi_v - 1 = a\phi_v, \quad \phi_v\psi_u - \phi_u\psi_v + 1 = b\phi_v, \quad \phi_u + \psi_v = c\phi_v,$$

By Fiedler's method of projection * a congruence of cycles corresponds to a real surface. Then the derivative of a polygenic function can be represented by a surface, which will be a plane when

$$(a + b)(\phi_v\psi_u - \phi_u\psi_v) + a - b + c(\phi_u + \psi_v) + d\phi_v = 0,$$

where a, b, c, d are constants.

We reword the above theorems as follows:

I′. The derivative surface of a polygenic function will be the plane $z = 0$ if the Jacobian of ϕ and ψ is equal to -1.

II′. The derivative surface will be the plane $x = 0$ if the Jacobian is equal to 1.

III′. The surface will be the plane $y = 0$ if $\phi_u = -\psi_v$.

IV′. The derivative will be represented by the Y axis if both conditions given in II′ and III′ are satisfied.

VI′. If the derivative surfaces of two polygenic functions are to be obtainable from one another by reflection in the XY plane, the conditions of VI must be satisfied.

VII′. The plane $x + z = 0$ represents the derivative of a polygenic function with vanishing Jacobian.

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* Fiedler, *Zyklographie*. If (A, B) are the Cartesian co-ordinates of the center and C the radius of a cycle, then $x = A$, $y = B$, $z = C$.

IX′. If in addition to the vanishing of the Jacobian, we have $\phi_u = -\psi_v$ the derivative is represented by the line

$$y = 0, \qquad z = -x.$$

X′. If the conditions of X are satisfied, the derivative is represented by the line

$$x = a, \qquad y = b.$$

XI′. The conditions of XI will give a point as the picture of the derivative.

We shall next examine the method of description of the derivative cycle.

From equation (2) for α it is evident that the change in α is proportional to the change in v', the ratio of the rates of change being equal to ϕ_v.

Let (θ, ρ) be Hessian co-ordinates of the ray (α, β). As v' varies, that is, as the point of approach moves along the given ray in the (u, v) plane, the point of contact of the ray (α, β) moves along the derivative cycle, and the ratio of the rates of the two points is expressed by

$$(1/C)\,dv'/d\theta,$$

where C has the value given above.

If we assume that the distribution of points on the ray in the (u, v) plane is uniform, then the absolute value of this ratio may be called the *density* of points on the derivative cycle. Representing this quantity by δ we have

$$\delta = \sec^2 \tfrac{1}{2}\theta/2 \,|\, \phi_v\psi_u - \phi_u\psi_v - 1 \,|\,. \tag{5}$$

Thus the distribution of points on the derivative cycle will be uniform only if θ is constant, which will occur only when the derivative cycle degenerates into a single ray or two points at infinity, (that is, if $\phi_v = 0$). If

$$\phi_v \neq 0 \quad \text{and} \quad \phi_u\psi_v - \phi_v\psi_u \neq -1,$$

then as v' increases from $-\infty$ to $-\phi_u/\phi_v$ to $+\infty$, the point of contact on the derivative cycle moves from the position $\theta = -\pi$ to $\theta = 0$ to $\theta = \pi$, and δ varies from ∞ to $(1/2) \,|\, \phi_v\psi_u - \phi_u\psi_v - 1 \,|$ to ∞, assuming equal values at points of the derivative cycle symmetrical with respect to the horizontal diameter.

2. *The Second Derivative.* Let $\sigma = \sigma_1 + j\sigma_2$ represent the second derivative of the polygenic function $W = \phi + j\psi$. Then

$$\sigma = (W_{uu} + 2v'W_{uv} + v'^2W_{vv})/(1 + jv')^2 + v''(W_v - jW_u)/(1 + jv')^3, \tag{6}$$

and hence

$$
\begin{aligned}
\sigma_1 &= \phi_{uu} + 2v'\phi_{uv} + v'^2\phi_{vv} + v''\phi_v, \\
\sigma_2 &= \psi_{uu} + 2v'(\psi_{uv} - \phi_{uu}) + v'^2(\psi_{vv} - 4\phi_{uv}) \\
&\qquad - 2v'^3\phi_{vv} + v''(\psi_v - \phi_u - 3v'\phi_v).
\end{aligned} \tag{7}
$$

The second derivative of a polygenic function will thus depend, in general, not only on the ray for which the derivative is taken, but on v' and v'' as well.

It will be independent of v'' only when

$$\phi_v = 0 \quad \text{and} \quad \phi_u = \psi_v,$$

that is, only when the function is monogenic.

The second derivative can be expressed as a function of v' and r, the radius of curvature, by the substitution

$$v'' = 2(r + uv' - v)/(1 + u^2)$$

in the above expression. We have

$$
\begin{aligned}
\sigma_1 &= \phi_{uu} - [2\phi_{vv}/(1 + u^2)] \\
&\quad + 2\{\phi_{uv} + [u\phi_v/(1 + u^2)]\}v' + \phi_{vv}v'^2 + [2\phi_v r/(1 + u^2)], \\
\sigma_2 &= \psi_{uu} + [2(\phi_u - \psi_v)v/(1 + u^2)] \\
&\quad + 2\{\psi_{uv} - \phi_{uu} + [u(\psi_v - \phi_u) + 3\phi_v v/(1 + u^2)]\}v' \\
&\quad + (\psi_{vv} - 4\phi_{uv} - [6u\phi_v/(1 + u^2)]v'^2 \\
&\quad - 2\phi_{vv}v'^3 + [2(\psi_v - \phi_u - 3v'\phi_v)r/(1 + u^2)].
\end{aligned} \tag{8}
$$

If we fix v' and allow r to vary, that is, if we fix a ray in the (u, v) plane, and a point on the ray, and allow a neighboring ray to approach the given ray along different tangent cycles, we find that the values which the second derivative assumes may be plotted as a *cycle*. For, if we eliminate r between the equations (8) we obtain

$$\sigma_2 = a + b\sigma_1, \tag{9}$$

where a and b are functions of u, v and the partial derivatives.

This is a cycle tangent to the reverse Y axis.

From the equations (8) we see that the change in σ_1 is proportional to the change in r. As r varies the cycle (9) is generated in a manner similar to that explained for the first derivative cycle above.

Now if we let v' vary we have ∞^1 cycles tangent to the reverse Y axis whose centers lie on a curve

$$x = \alpha y^3 + \beta y^2 + \gamma y + \delta,$$

where the coefficients depend on the partial derivatives.

If the second partial derivatives all vanish, that is, if

$$W = Au + Bv + C \qquad (A, B, C \text{ are dual numbers})$$

the ∞^1 cycles become all the points of the Y axis.

If in (8) we hold r constant and allow v' to vary, that is, if we consider the ∞^1 values of the second derivative for a given ray as this ray is approached along elements of the same curvature, then we have as the picture of the second derivative the curve represented by (8) if we regard v' as parameter.

In general, this curve (non-oriented) is one of the *sixth* order. If $\phi_{vv} = 0$, this curve becomes a *cycle*.

3. *Linear Fractional Polygenic Functions.* By a linear fractional polygenic transformation we shall mean one of the set of ∞^{10} transformations

$$W = (Aw + B\bar{w} + C)/Dw + E\bar{w} + F), \tag{10}$$

where the constants are dual numbers.

This set of transformations is of interest, since it includes the Laguerre Group as a sub-group. It has also the sub-group

(11)
$$U = (au + bv + c)/(du + ev + f),$$
$$V = (gu + hv + k)/(du + ev + f).$$

It will reduce to the former group if

$$B = E = 0 \quad \text{or} \quad A = D = 0.$$

It will reduce to the latter if

$$\bar{D}/E = \bar{E}/D = \bar{F}/F.$$

The group (11) will transform cycles tangent to the reverse y-axis into cycles, but will transform other cycles into curves of higher order.

The set (10) also includes the group

$$W = aw + b\bar{w} + c, \tag{12}$$

which converts cycles into cycles.

All invariants under (12) are combinations of

$$v^{(n)}(v'')^{n-3}/(v''')^{n-2}, \qquad \text{where } n \geqq 4.$$

Further discussion of these transformations and invariants will be published elsewhere.

Trinomial Curves and Monomial Groups.*

By R. M. WINGER.

1. *Introduction.* Maschke † has considered a particular class of monomial collineation groups in the ternary domain, namely groups generated by the two substitutions

$$x' = y, \quad y' = z, \quad z' = x,$$
$$x' = ax, \quad y' = by, \quad z' = cz,$$

where a, b, c are roots of unity and $abc = 1$. These restrictions exclude even monomial groups requiring more than two generators, as well as those with two generators one of which is a homology whose period is a multiple of 3. Furthermore any of Maschke's groups can be enlarged by adding as a generator a homology (of period greater than 2) having a vertex and opposite side of the fixed triangle for center and axis. Skinner ‡ has extended the study to a larger class of monomial groups, viz. those containing only elements of determinant ± 1.

Both authors, however, exclude the interesting groups that leave invariant the trinomial curves $x^n + y^n + z^n = 0$. These curves have received attention at the hands of numerous writers.§ But the group aspect of the curves has been neglected except for a few particular cases. From this point of view Berzolari ¶ has treated the projective lemniscate ($n = -2$) whose associated group is the ternary octahedral G_{24}. The quartic case has been studied by Dyck.‖ Snyder ** notices the case $n = 5$ in his study of quintic curves invariant under linear transformations. Tappan †† gives the generators of the group for the sextic, while Musselman ‡‡ discusses the group and curve briefly. Finally the author ¶¶ has discussed in considerable detail

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* Read before the American Mathematical Society, December 28, 1927.

† *American Journal of Mathematics*, Vol. 17 (1895), p. 168.

‡ *Ibid.*, Vol. 25 (1903), p. 17.

§ For references, see Loria, *Spezielle Algebraische und Transzendente Ebenen Kurven*, Vol. 1, p. 328 ff.

¶ *Istituto Lombardo, Rendiconti*, Ser. 2, Vol. 37 (1904), pp. 277, 304.

‖ *Mathematische Annalen*, Vol. 17 (1880), p. 510.

** *American Journal of Mathematics*, Vol. 30 (1908), p. 7.

†† *Ibid.*, Vol. 37 (1915), p. 320.

‡‡ *Ibid.*, Vol. 49 (1927), p. 355.

¶¶ *Tôhoku Mathematical Journal*, Vol. 29 (1928), pp. 376-400. Muth, *Dissertation*, Giessen (1890), devotes some space to the equianharmonic cubic, $n = 3$.

the cases $n = \pm 3$ which are invariants of the monomial subgroup G_{54} of the ternary Hesse group. The generalization of the principal results of that paper is the present objective.

The purpose of the paper is thus twofold: the analysis of the group and an investigation of trinomial curves for an arbitrary integral value of n. The invariant configuration of the group is described and the relation of invariant curves thereto is pictured. The complete system of invariants of the group is obtained and several constructions of conjugate sets of points on individual curves of the systems are explained. A quadratic inversion is introduced which proves to be an effective auxilliary weapon to the collineation group. In conclusion some generalizations to n-space are sketched.

Broadly speaking we find that the geometry of the group and the invariant curves differs according as n is odd or even and again according to the classification of n in the residue system with respect to the modulus 3. This provides a natural basis for subdividing the curves into species.

2. *The Structure of the Group.* If we consider the substitutions on the variables which leave unaltered the curve

$$x^n + y^n + z^n = 0, \qquad n \text{ a positive integer},$$

they must be either permutations of the letters or permutations of the letters multiplied by an nth root of unity.* All such substitutions give rise to collineations which can be generated by the following set

$$\begin{array}{rccccl} & S & T & R & W & \\ x' = & \epsilon x & z & y & \epsilon x & \\ y' = & \epsilon^{-1} y & x & x & y & \\ z' = & z & y & z & z, & \epsilon = e^{2\pi i/n}. \end{array}$$

For T and R in combination effect all permutations of the letters, while products of powers of S and W have the effect of multiplying the letters by the nth roots of unity in all possible ways.

We suppose throughout that the variables are homogeneous projective co-ordinates, that the transformed variables are primed, and that multiplication if from left to right. The vertices of the triangle of reference opposite the sides x, y, z, respectively, we shall denote by u, v, w. We shall at pleasure refer to the individual collineations of the group as "elements." An element of period two, i. e. a harmonic homology, we shall call a "reflexion."

We observe first that *the order of the group is* $6n^2$. For there are n^2

* There is an exception in the case of the conic which may admit still other types of substitutions.

projectively distinct collineations which do not disturb the order of the variables, viz.

$$(1) \qquad x' = x, \quad y' = \epsilon^i y, \quad z' = \epsilon^j z, \qquad i, j, = 1, 2, \cdots, n.$$

These n^2 elements form a subgroup whose generators are S and W. Now T and R generate a dihedral subgroup of order 6, the products of whose elements by the n^2 elements (1) aggregate a total of the $6n^2$ projectively distinct collineations. It follows that the elements of the group fall into six sets of n^2 each, corresponding to the six permutations of the variables. Or,

The group is in $(n^2, 1)$ *isomorphism with* (a) *the permutation group on three letters,* (b) *the dihedral group of order* 6 *generated by* T *and* R.

The dihedral G_6 is thus a factor group of the whole group. The invariant subgroup corresponding to the identical element of the factor group is that generated by S and W. This group is also Abelian since it comprises all those elements which do not permute the letters. Henceforth we shall denote the Abelian invariant subgroup, consisting of the elements (1), by G. Further by GT, for example, we shall mean the n^2 elements formed by multiplying each element of G by T. Then the six sets of n^2 elements of our group may be represented symbolically as follows:

$$(2) \qquad \begin{matrix} G & GT & GT^2 \\ GR & GTR & GT^2R \end{matrix}$$

or

$$(3) \qquad \begin{matrix} S^iW^k & S^iW^kT & S^iW^kT^2 \\ S^iW^kR & S^iW^kTR & S^iW^kT^2R, \end{matrix} \qquad (i, k = 1, \cdots, n).$$

To find the element of the factor group to which the n^2 elements represented by any symbol in (2) correspond, we have merely to replace the "G" in that symbol by 1. Or if we replace by 1 the multipliers of any collineation of the entire group we shall obtain that element of the factor group to which the collineation corresponds.

The triangle of reference obviously is fixed under the group: all the elements of the subgroup G leave each vertex fixed; the elements GT and GT^2 permute the vertices cyclically while the remaining elements leave one vertex fixed and interchange the other two.

The group G evidently contains two homologies with multipliers 1, ϵ, 1 and 1, 1, ϵ, and each of them as well as W generates a cyclic subgroup of order n, consisting wholly of homologies. Hence

G contains three sets of $n-1$ homologies, each set having a vertex and opposite side of the reference triangle for center and axis.

These homologies,—and indeed all elements of G, excepting the identity, are of period n if n is prime. When n is even there is one reflexion in each set of $n-1$.

Again, S and R generate a dihedral subgroup of order $2n$. The n reflexions of this group belong to the set GR. Their centers lie on the z-axis and their axes are concurrent at the opposite vertex. Therefore,

The group includes three dihedral subgroups of order $2n$, each having a vertex and opposite side of the fixed triangle as invariant point-line. Further the group contains $3n$ reflexions whose centers lie in sets of n on the sides of the fixed triangle and whose axes, in sets of n, meet at the vertices.

There is a difference in the dihedral groups according as n is odd or even. When n is odd, the axes of reflexion are conjugate and the centers do not lie on the axes. On the other hand when n is even, the axes divide into two conjugate sets, say ξ_1, and ξ_2 of $n/2$. And each center lies on an axis: the centers associated with ξ_i lie on the axes ξ_j when $n/2$ is odd, and on ξ_i when $n/2$ is even. In any case the equation of the axes of reflexion is

$$(y^n - z^n)(z^n - x^n)(x^n - y^n) = 0$$

and the co-ordinates of the centers are

$$(0, \epsilon^i, -1), \quad (\epsilon^i, -1, 0), \quad (-1, 0, \epsilon^i).$$

The multiple isomorphism implies the existence of several subgroups, corresponding to the subgroups of the factor group. Thus the elements of the first row of the table (2) (or (3)) constitute a second invariant subgroup, generated by S, T, W, corresponding to the cyclic G_3 of the factor group. This group is of order $3n^2$ and comprises those elements which effect the cyclic advance (xyz) of the variables, multiplied by roots of unity. It is easily verified that the elements of this group, not contained in G, are all of period three and that the square of each element GT is an element GT^2 and vice versa. Hence

The group contains $2n^2$ elements of period 3, which belong in n^2 cyclic subgroups G_3.

These subgroups, as we shall prove in the next section, are all conjugate when n is not a multiple of 3. When n is a multiple of 3, they form two conjugate sets, containing respectively one-third and two-thirds of the groups.

We have now accounted for half the elements of the group. We have also noted $3n$ reflexions belonging to the other half GT^1R. The remaining elements of this second-half are all of even period since they correspond to

elements of period two in the factor group, but the precise periods depend on the factorability of n.

The elements corresponding to each of the reflexions of the factor group, together with the elements of G, form a subgroup of order $2n^2$. Thus the elements GR, corresponding to R, are S^iW^kR, $i, k = 1, \cdots, n$. The square of each of these is some homology of G. For example the collineation S^iW^kR and its square are respectively

$$x' = \epsilon^{-i}y, \qquad y' = \epsilon^{i+k}x, \qquad z' = z. \tag{4}$$

$$x' = \epsilon^{k}x, \qquad y' = \epsilon^{k}y, \qquad z' = z. \tag{5}$$

When $k = n$, (4) represents the $3n$ reflexions of the dihedral G_{2n} generated by S and R.

When $k = 1$, the homology (5) is of period n whence (4) is of period $2n$. If now p of the nth roots of unity are primitive, then the number of collineations of period $2n$ in GR is np, for with each value of i $(= 1, \cdots, n)$ can be combined p values of k. Each collineation of period $2n$ generates a cyclic subgroup G_{2n}, of order $2n$. The n even powers of a generator of such a subgroup are homologies belonging to G, but the odd powers are all included in the set GR. The number of cyclic G_{2n}'s, however, depends on the nature of n.

n, odd. Each G_{2n} contains p elements of period $2n$, since there are p primitive $2n$-th roots of unity. Accordingly there are n cyclic G_{2n}'s generated by the elements of period $2n$ in GR. The odd powers of these generators are all distinct and comprise all the elements GR. Similar statements hold for the sets GTR and GT^2R. Hence

The group contains $3n$ cyclic subgroups G_{2n}, of order $2n$ when n is odd. Half the elements of these groups are homologies (including the identity) from G while the other half comprise all the $3n^2$ elements corresponding to the elements of period two in the factor group.

The nth power of any generator of each of these G_{2n}'s is one of the $3n$ reflexions.

n, even. Since there are now $2p$ primitive $2n$-th roots, each cyclic G_{2n} contains $2p$ elements of period $2n$. Accordingly, there are $3n/2$ cyclic G_{2n}'s whose elements, not included in G, comprise but half of the $3n^2$ elements corresponding to the elements of period two in the factor group. The remaining half will contain elements of period n, $-e$, g, S^iW^kR when ϵ^k, is a primitive $n/2$-th root of unity. If now $n/2$ is odd, GR will contain np elements of period n, and a cyclic G_n will include p of them. Hence we

shall have in the whole group $3n$ cyclic subgroups of order n, whose elements (not contained in G) comprise the remaining half of the $3n^2$ elements GT^jR, $j = 1, 2, 3$. But if $n/2$ is even and $n/4$ is odd, there will be $np/2$ elements of period n in GR, p of which will be contained in each G_n. The whole group then will possess $3n/2$ cyclic G_n's, whose elements (exclusive of those in G) comprise one-fourth of the elements GT^jR. The remaining fourth of these elements are distributed equally in $3n$ cyclic groups of order $n/2$. For there are $3np/2$ elements in GT^jR of period $n/2$, and $p/2$ of these occur in each $G_{n/2}$. And so for any even n—we remove the factor 2 repeatedly until the remaining factor is odd. Hence

If n is even and $n/2^a$ is the first odd number in the sequence n, $n/2$, $n/2^2, \cdots$, the $3n^2$ elements GT^jR are distributed as follows:

$3n^2/2$ *among* $3n/2$ *cyclic groups of order* $2n$;
$3n^2/2^2$ *among* $3n/2$ *cyclic groups of order* n;
$3n^2/2^3$ *among* $3n/2$ *cyclic groups of order* $n/2$;

. . . . . .

$3n^2/2^a$ *among* $3n/2$ *cyclic groups of order* $n/2^{a-2}$;
$3n^2/2^a$ *among* $3n$ *cyclic groups of order* $n/2^{a-1}$.

Half of the elements of each of these cyclic subgroups (the even powers of the generators) are homologies from G, while the odd powers of the generators are elements GT^jR. Each of the $3n$ cyclic groups of order $n/2^{a-1}$ contains one of the $3n$ reflexions. Note that the number of groups of the several orders is equal except the last which is double that of the others. In particular, if $n = 2^a$, the last set of groups are the $3n$ cyclic groups of order two generated by the $3n$ reflexions.

3. *Special Sets.* The special sets of conjugate points and lines under the group are compounded of the fixed points and lines of the individual collineations, presenting themselves in dual configurations. Now every element of G leaves unaltered each vertex and side of the triangle of reference, while each homology of course leaves unaltered every point on one side of the triangle.

The fixed points of the n^2 cyclic G_3's, comprising (besides the identity) the elements GT^j $(j = 1, 2)$ aggregate $3n^2$. The axes of reflexion meet by threes at n^2 of these points,—aside from the vertices of the reference triangle. Thus for all values of $i, j = 1, \cdots, n$ the point $(\epsilon^i, \epsilon^j, 1)$ lies on the three axes.

$$y - \epsilon^j z = 0, \qquad \epsilon^i z - x = 0, \qquad \epsilon^j x - \epsilon^i y = 0.$$

And obviously only one axis from each vertex can contain any particular point. When $i = j$, which happens n times, the point lies on the axis $x - y = 0$. Evidently the axes are treated symmetrically. Hence, neglecting the intersections at the vertices of the reference triangle,

The $3n$ axes of reflexion meet by threes at n^2 points, n of the points lying on each axis.

Each of the n^2 points is the fixed point of a dihedral subgroup G_6.

For in addition to a cyclic G_3, each is fixed under three reflexions the product of any two of which generate the cyclic G_3. The three centers of these reflexions lie on a line, the fixed line of the dihedral G_6. The n^2 fixed lines of the dihedral groups are dual to the n^2 fixed points while the centers of reflexion are dual to the axes, hence

The $3n$ centers of reflexion lie in sets of n on the sides of the reference triangle and by threes on n^2 lines, n of the lines meeting at each center.

The generator of the cyclic subgroup G_3 of the dihedral G_6 which has the fixed point $(\epsilon^i, \epsilon^j, 1)$ may be written

$$x' = \epsilon^{2i-j}y, \qquad y' = \epsilon^{i+j}z, \qquad z' = x. \tag{6}$$

The two additional fixed points of the G_3 are

$$(\omega\epsilon^i, \omega^2\epsilon^j, 1) \text{ and } (\omega^2\epsilon^i, \omega\epsilon^j, 1), \qquad \omega^3 = 1. \tag{7}$$

If n is not a multiple of 3 the co-ordinates (other than 1) of these points are $3n$-th roots of unity but not n-th roots. And if i, j range over the values of $1, \cdots, n$, we get n^2 cyclic G_3's, each of which has for fixed points one of the n^2 points, and two of a second set of $2n^2$ points. Furthermore, it is obvious from the form of the co-ordinates that both aggregates of points are (special) conjugate sets. If, however, n is a multiple of 3, we get only $n^2/3$ cyclic G_3's by varying i and j, whose fixed points comprise the n^2 points. In this case we have three dihedral G_6's, one associated with each vertex of a triangle of fixed points, and having a common invariant subgroup G_3.

In any case the generators of the n^2 cyclic G_3's may be written

$$x' = \epsilon^i y, \qquad y' = \epsilon^j z, \qquad z' = x, \qquad i, j = 1, \cdots, n. \tag{8}$$

The fixed points of the groups are

$$(\eta^{i+j}, \eta^{2j-i}, 1), \quad (\omega\eta^{i+j}, \omega^2\eta^{2j-i}, 1), \quad (\omega^2\eta^{i+j}, \omega\eta^{2j-i}, 1), \qquad \eta = e^{2\pi i/3n}. \tag{9}$$

The junction of any two is the polar of the third with respect to the reference triangle, or

The fixed points and lines of each of the n^2 cyclic G_3's form a self-polar triangle with respect to the invariant triangle of the whole group.

Suppose now that $n \equiv 0$, mod 3. Then if $i + j \equiv 0$, mod 3, (8) represents the $n^2/3$ cyclic G_3's whose fixed points comprise the set of n^2 points. If $i + j \not\equiv 0$, mod 3, (8) represents the $2n^2/3$ cyclic G_3's whose fixed points aggregate $2n^2$. The co-ordinates of the $2n^2$ fixed points as a set may be written

$$(10) \qquad (\eta^{3k+1}, \eta^{3k-1}, 1), \qquad (\eta^{3k-1}, \eta^{3k+1}, 1),$$

where k is an integer, there being n^2 points represented by each expression. All these numbers η^a are $3n$-th roots of unity which are not n-th roots. The co-ordinates of the $2n^2$ points may thus be written in the form

$$(11) \qquad (\epsilon^i\eta, \epsilon^j\eta^2, 1), \qquad (\epsilon^i\eta^2, \epsilon^j\eta, 1), \qquad i, j = 1, \cdots, n,$$

from which it is apparent that they constitute a special conjugate set. We may summarize as follows:

The $3n^2$ fixed points of the n^2 cyclic subgroups G_3 divide into two special conjugate sets of n^2 and $2n^2$ points respectively. If $n \not\equiv 0$, mod 3, the cyclic groups are conjugate, each having for fixed points one of the former and two of the latter. But if $n \equiv 0$, mod 3, the groups themselves divide, one-third and two-thirds into conjugate sets, the fixed points of one being the n^2 points and those of the other the $2n^2$ points. The $3n^2$ fixed lines form dual configurations.

The square of each element GT^jR, $j = 1, 2, 3$, is a homology of G (or the identity). The fixed points of each of the cyclic groups G_r $(r > 2)$ generated by them, accordingly, are a vertex of the triangle of reference and two points on the opposite axis. For example, the fixed points of the group generated by S^iW^kR [equations (4)] which lie on the z-axis are $(\epsilon^{-i}, \pm\epsilon^{k/2}, 0)$. These obviously are cut out by the lines $x^{2n} - y^{2n} = 0$. But half of the points are centers of the n reflexions included among the elements in question. The centers lie on the lines $x^n + y^n = 0$ when n is odd, on $x^n - y^n = 0$ when n is even. Since the sides of the triangle of reference are treated symmetrically, we conclude:

*Each element GT^jR, $j = 1, 2, 3$, leaves fixed one vertex of the invariant triangle of the group. The residual fixed points of the elements make up two special sets of $3n$ conjugate points, lying on the sides of the invariant triangle and cut out by the two special dual sets of conjugate lines,**

* Dual to the sets of points not to each other.

$$(y^n + z^n)(z^n + x^n)(x^n + y^n) = 0, \quad (y^n - z^n)(z^n - x^n)(x^n - y^n) = 0.$$

One set of $3n$ points are centers of reflexion, which are cut out by the first set of lines when n is odd, by the second set of lines (axes) when n is even. This completes the enumeration of special sets of conjugate points—except the obvious sets on the axes of homology and reflexion—which we collect for convenience of reference.

1°. One set of 3, vertices of the invariant triangle;

2°. A set of $3n$, centers of reflexion, cut out of the sides of the invariant triangle by $\Pi(x^n + y^n) = 0$, when n is odd, by $\Pi(x^n - y^n) = 0$, when n is even; *

3°. A counter set of $3n$, cut from the sides of the invariant triangle by $\Pi(x^n - y^n) = 0$, when n is odd, by $\Pi(x^n + y^n) = 0$ when n is even;

4°. One set of n^2, intersections of the axes of reflexion aside from the vertices of the invariant triangle;

5°. One set of $2n^2$, comprising with the n^2 points the fixed points of the n^2 cyclic G_3's;

6°. ∞^1 sets of $6n$, lying on the sides of the invariant triangle, each fixed under a homology of period n;

7°. ∞^1 sets of $3n^2$, lying on the axes of reflexion.

The duals of these are the special sets of conjugate lines, all other points and lines belonging to general conjugate sets.

4. *The Complete System of Invariants.* By the same argument used in the case of the cubic † it is proved that the complete system of invariants of our group consists of the four forms

$$\begin{aligned} s_1 &: \quad x^n + y^n + z^n \\ s_2 &: \quad y^n z^n + z^n x^n + x^n y^n \\ \Pi_3 &: \quad x\,y\,z \\ \Pi_{3n} &: \quad (y^n - z^n)(z^n - x^n)(x^n - y^n). \end{aligned}$$

These forms are connected by the syzygy

$$(12) \qquad -(\Pi_{3n})^2 \equiv 27 s_3^2 + 4 s_2^3 + 4 s_1^3 s_3 - s_1^2 s_2^2 - 18 s_1 s_2 s_3,$$

where $s_3 = x^n y^n z^n$. Another obvious invariant is

$$\Pi'_{3n}: (y^n + z^n)(z^n + x^n)(x^n + y^n) \equiv s_1 s_2 - s_3.$$

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* By $\Pi(x^n + y^n)$ we mean the product of three terms of the type within the parentheses.

† Winger, *Tôhoku Mathematical Journal*, Vol. 29 (1928), § 3.

These forms are associated with the covariant system of the binary cubic whose roots are x^n, y^n, z^n:

$$t^3 - s_1 t^2 + s_2 t - s_3. \tag{13}$$

Thus $-(\Pi_{3n})^2$, being the square of the product of differences of the roots, is the discriminant Δ of the cubic,—a fact which leads at once to the syzygy (12).

There is an interesting connection between (a) the covariant system of the cubic, (b) the invariants of the group, and (c) the invariants of the inversion

$$x' = 1/x, \qquad y' = 1/y, \qquad z' = 1/z. \tag{14}$$

The Hessian H and cubicovariant $C_{3,3}$ of the cubic are respectively

$$H: \ (3s_2 - s_1^2)t^2 + (s_1 s_2 - 9s_3)t + (3s_1 s_3 - s_2^2),$$

$$C_{3,3}: \ (-27s_3 + 9s_1 s_2 - 2s_1^3)t^3 + (9s_1 s_3 + s_1^2 s_2 - 6s_2^2)t^2 + (-6s_1^2 s_3 + 9s_2 s_3 + s_1 s_2^2)t + (9s_1 s_2 s_3 - 27s_3^2 - 2s_2^3).$$

The binary inversion $t' = 1/t$ has the same effect on the cubic and its covariants as the ternary inversion (14): the coefficients are interchanged in pairs, first with last, second with next to last, etc. Thus *the fundamental forms s_1 and s_2 are interchanged while the systems of lines Π_{3n} and Π_{3n}' and the pencil of curves $s_1 s_2 - \lambda s_3$ are invariant.*

The inversion (14) also leaves invariant as a whole the special set of n^2 points and interchanges in pairs the set of $2n^2$ points (10).

We shall now take up the properties of the fundamental invariants. First we observe that *the curve s_1 has n hyperosculation points on each side of the invariant triangle, whose n tangents meet at the opposite vertex. The points are the centers of reflexion when n is odd and the counter sets of 3n when n is even. In both cases the equation of the 3n tangents is $\Pi_{3n}' = 0$.*

The hyperosculation points absorb all of the flexes and $\frac{3}{2}n(n-2)(n-3)$ double lines. There remain $\frac{1}{2}n^2(n-2)(n-3)$ double lines. Again

Each of the 3n axes of reflexion cuts the curve in n points whose (simple) tangents meet at the corresponding center.

The points and lines are special conjugate sets of $3n^2$. Finally

The $2n^2$ points are the complete intersections of the fundamental invariants s_1 and s_2.

This is proved either by solving simultaneously the equations of the

curves or by substituting the co-ordinates (10) of the points already found. We have found three special sets of points on the curve, all other points occurring in general sets.

We shall now prove two general theorems on curves admitting homologies.

1°. *If a curve is invariant under a homology of period k, the rth polar ($r < k$) of the center of the homology is composite, containing the axis as a $(k-r)$-fold factor.*

Let the homology be

$$x' = x, \qquad y' = y, \qquad z' = \theta z, \qquad \theta^k = 1.$$

Then the equation of the curve is of the form

$$f_0 + f_1 z^k + f_2 z^{2k} + f_3 z^{3k} \cdots = 0,$$

where f_i are binary forms in x and y. The rth polar of the center (0, 0, 1) of the homology contains the factor z^{k-r}.

2°. *If a curve of order n admits a homology of period n, the first polar of any point on the axis of homology breaks up into $n-1$ lines which meet at the center.*

For the multiplier of z in the homology is ϵ, hence the equation of the curve may be written

$$z^n + f^{(n)}(x, y) = 0,$$

where $f^{(n)}$ is a binary form of order n. And the polar of any point $(a, b, 0)$ on the axis of homology is a binary form in x and y of order $n-1$, which represents a complex of $n-1$ lines on the center.

Theorem 2°, applied to the curve s_1, yields the theorem:

*The contacts of tangents from an arbitrary point on a side of the invariant triangle fall in sets of n on $n-1$ lines which meet at the opposite vertex.**

From the inverse relation of s_1 and s_2 we infer at once:

Each vertex of the invariant triangle is for the curve s_2 an n-fold point, at each branch of which the tangent has $(n+1)$-point contact. The tangents at the multiple points are $\Pi_{3n}' = 0$.

These points, because of their multiple tangents, absorb $3n(n-1)$ flexes

* From each hyperosculation point however can be drawn only $n^2 - 2n$ tangents (in addition to the tangent at the point) which lie on $n-2$ lines through the opposite vertex.

and $3n(n-1)(n-2)/2$ double lines in addition to the normal quota. Accordingly the Plücker numbers of the curve are as follows:

Order, $2n$; Class, $n(n+1)$; Genus, $\frac{1}{2}(n-1)(n-2)$; Number of flexes, 0 *; Number of double lines, $\frac{1}{2}n^2(n-1)$.

The n intersections of each axis of reflexion, aside from those at a multiple point, are contacts of simple tangents from the corresponding center, the aggregate of such points forming a special conjugate set of $3n^2$.

The curve s_2 like s_1 has but three special sets of points, including the common set of $2n^2$ points. The $2n^2$ points deserve special attention. We distinguish three cases according as the residue of n with respect to the modulus 3 is 0, 1 or 2.

Suppose first that n is of the form $3k$. Then we saw, § 3, that the $2n^2$ points comprise the fixed points (9) of the $2n^2/3$ cyclic groups (8), $i+j \not\equiv 0$, mod 3. Under the restrictions imposed we find that the three points (9) are the vertices of a Poncelet triangle alike for the curves s_1 and s_2. In fact, denoting the points in the order written by 1, 2, 3, and by (123) indicating that the tangents at the points respectively pass through 2, 3, 1, then the symbol for the triangle on s_1 is (123) when $i+j$ has the form $3k+1$, and (132) when $i+j$ has the form $3k+2$. On s_2 the symbols for the triangle are interchanged under the respective hypotheses. The sides of the triangles are naturally the special sets of $2n^2$ lines on the two curves. Thus

If n is a multiple of 3, the curves s_1 and s_2 have a common system of $2n^2/3$ Poncelet triangles, each the fixed triangle of a cyclic G_3, whose vertices and sides in the aggregate comprise respectively the special sets of $2n^2$ conjugate points and lines.

If n is not a multiple of 3, the fixed points of the n^2 cyclic G_3's are

$$(\epsilon^i, \epsilon^j, 1), \qquad (\omega\epsilon^i, \omega^2\epsilon^j, 1), \qquad (\omega^2\epsilon^i, \omega\epsilon^j, 1) \qquad i, j = 1 \cdots n.$$

For a given i, j, the first is an n^2 point while the other two belong to the set of $2n^2$ points. Again, the first is the unique fixed point and the others a conjugate pair of the allied dihedral G_6. The tangents to s_1 at $(\omega\epsilon^i, \omega^2\epsilon^j, 1)$ and $(\omega^2\epsilon^i, \omega\epsilon^j, 1)$ are respectively $\omega^{n-1}\epsilon^{-i}x + \omega^{2n-2}\epsilon^{-j}y + z = 0$ and $\omega^{2n-2}\epsilon^{-i}x + \omega^{n-1}\epsilon^{-j}y + z = 0$. These pass through $(\epsilon^i, \epsilon^j, 1)$ if $n = 3k+2$. But if $n = 3k+1$, they both reduce to

* Except when $n = 2$. The curve is then the projective lemniscate, having three biflecnodes. When n is an even number greater than 2, the branches at the multiple points are sometimes called flexes but properly speaking they are higher singularities.

(15) $$\epsilon^{-i}x + \epsilon^{-j}y + z = 0,$$

which is thus a double line. This situation is just reversed for the curve s_2: (15) is a double line when $n = 3k + 2$; but if $n = 3k + 1$, the tangents at the $2n^2$ points meet in pairs at the n^2 points. Hence, the n^2 cyclic G_3's being conjugate,

If n is of the form $3k + 1$, the special set of n^2 lines (15) *are double lines of s_1 with contacts at the special set of $2n^2$ points (cut out by s_2). The special set of $2n^2$ lines are tangent to s_2 at the $2n^2$ points and meet in pairs at the n^2 points,—the polars of the n^2 lines with respect to the invariant triangle of the group. When n is of the form $3k + 2$, the rôles of s_1 and s_2 are interchanged.*

We are now in a position to enumerate the double lines which meet at the centers of reflexions, the tangents from a center being either simple, with contacts on the axis, or double lines.

The double lines of s_1. n, odd. Each center is a hyperosculation point, whose tangent is equivalent to n simple tangents. There are also n simple tangents with contacts on the corresponding axis. The remaining tangents combine into $(n^2 - 3n)/2$ double lines. When n is of the form $3k$ or $3k + 2$, these double lines are all distinct. We have then accounted for $\frac{3}{2}n^2(n-3)$ of the double lines, leaving $\frac{1}{2}n^2(n-3)(n-5)$. If n is of the form $3k + 1$, we have just seen that the n^2 lines (15) are double lines. And since each contains three centers of reflexion, each has been counted three times in the above enumeration. Thus while the number of double lines meeting at each center is not affected, the total number involved is reduced to $\frac{1}{2}n^2(3n - 13)$, leaving $\frac{1}{2}n^2(n^2 - 8n + 19)$ unaccounted for.

n, even. The centers are not on the curve, so each carries n simple tangents with contacts on the axis, and $\frac{1}{2}n(n-2)$ double lines in addition. Again when n is of the form $3k$ or $3k + 2$, these double lines are all different so that $\frac{1}{2}n^2(n-2)(n-6)$ remain. But if n is of the form $3k + 1$, the n^2 lines are double lines and each is counted three times, leaving $\frac{1}{2}n^2(n-4)^2$ unaccounted for.

We have accounted for all the double tangents of the quartic, quintic and sextic cases. For higher values of n, those which remain fall into general conjugate sets, the number under the appropriate hypotheses in each case, being a multiple of $6n^2$. Finally, in consequence of theorem 2° of this section, we may say

All of the double lines of the curve meet in sets of n on the sides of the

invariant triangle of the group, while the contacts of each set divide into two sets of n lying on lines through the opposite vertices.

The double lines of s_2. n, odd. The tangent at each branch of a multiple point passes through a center of reflexion and is equivalent to n simple tangents from the center. From each center also run n simple tangents with contacts on the axis. The remaining tangents unite to form double lines. If n is of the form $3k$ or $3k+1$, there remain $\frac{1}{2}n^3(n-1) - \frac{3}{2}n^2(n-1) = \frac{1}{2}n^2(n-1)(n-3)$. If, however, n is of the form $3k+2$, the n^2 lines are double lines and each is counted three times since it contains three of the centers. In this case the number of residual double lines is $\frac{1}{2}n^2(n^2-4n+7)$.

n, even. The tangents at the multiple points are not on the centers of reflexion. Thus each center carries n simple tangents with contacts on its axis, and in addition $\frac{1}{2}n^2$ double lines. If $n = 3k$ or $3k+1$, there remain $\frac{1}{2}n^3(n-4)$. But if $n = 3k+2$, on removing duplicates occasioned by the triple counting of the n^2 lines, we are left with $\frac{1}{2}n^2(n-2)^2$.

We have completely accounted for the double lines of s_2 when $n = 2, 3, 4$. In all other cases, the residual double lines belong in general conjugate sets, the formula for the number in each case representing a multiple of $6n^2$.

5. *The Construction of Conjugate Sets. The Configuration of a General Set of 6n Conjugate Points.* An arbitrary point P of the plane is carried by the homology $x' = x$, $y' = y$, $z' = \epsilon z$, into n points of a line through the center w. Let this line be $\sqrt[n]{a}x + y = 0$. The homology $x' = x$, $y' = \epsilon y$, $z' = z$ will carry the line into a set of n lines, $ax^n + y^n = 0$, each of which contains n points conjugate to P. These lines in turn are transformed by the reflexion $x' = y$, $y' = x$, $z' = z$, into a second set of n, $x^n + ay^n = 0$, each containing n points conjugate to P. Now by a similar process we get two sets of lines from each of the other vertices, each line containing n points conjugate to P. We thus have $6n$ lines, $2n$ on each vertex of the reference triangle, and each carries n points—aggregating $6n^2$—which form a general conjugate set under the group.

Since these lines as a whole are invariant under the group they must be expressible in terms of the complete system. Such a set of $6n$ lines, which we shall call Λ_a, is

$$(16)\quad \begin{aligned}\Lambda_a \equiv{}& (ay^n + z^n)(y^n + az^n)(az^n + x^n)(z^n + ax^n)(ax^n + y^n)(x^n + ay^n)\\ \equiv{}& a^3(s_1^2 s_2^2 - 2s_1^3 s_3 - 2s_2^3 + 4s_1 s_2 s_3 - s_3^2)\\ &+ a^2(a^2+1)(s_2^3 + 3s_3^2 + s_1^3 s_3 - 5s_1 s_2 s_3)\\ &+ a(a^2+1)^2(s_1 s_2 s_3 - 3s_3^2) + (a^2+1)^3 s_3^2 = 0.\end{aligned}$$

For an arbitrary value of the parameter a, the lines of the system are distinct. But when $a = -1$, the lines are the axes of reflexion repeated and Λ_a reduces to $-(\Pi_{3n})^2$ or Δ. Again when $a = 1$, (16) represents the counter set of $3n$ lines repeated, or $\Lambda_1 \equiv (\Pi_{3n}')^2$.

We have not yet fully described the configuration of a general set of conjugate points. The n^2 points on the lines $ay^n + z^n = 0$ are invariant as a whole under the two homologies above, hence they lie in sets of n on n lines through each of the vertices v and w of the triangle of reference. Thus we have *a set of n^2 points and $3n$ lines, n lines meeting at each vertex. Each line contains n of the points and each point lies on three of the lines. The points therefore comprise the complete intersections of the lines aside from the vertices of the reference triangle. A general conjugate set consists of six such sets of n^2 points, one set on the lines corresponding to each factor in* (16).

The system of lines Λ_a mutually intersect in $12n^2$ points which in general belong in two special sets of $3n^2$, lying on the system Λ_{-a^2}. Exceptions occur when $a = 1$, or -1 as already noted and also when $a = -\omega$ (or $-\omega^2$), as will appear later.

Λ_a is also an invariant of the inversion (14). Indeed under the inversion each term (function of s_i) in (16) goes into itself. Each, however, can be built up out of the simpler invariants of the inversion, $s_2^3 + s_1^3 s_3 = 0$, and the members of the pencil $s_1 s_2 + \lambda s_3 = 0$.

We shall now consider the problem of the construction of sets of conjugate points on the fundamental invariants s_1 and s_2, utilizing the principle that invariant curves meet in sets of conjugate points.

A second composite invariant is

$$(17) \qquad \begin{aligned} K_c &\equiv (cy^n z^n + x^{2n})(cz^n x^n + y^{2n})(cx^n y^n + z^{2n}) \\ &\equiv cs_2^3 + (c+1)^3 s_3^2 + cs_1 s_3 [cs_1^2 - 3(c+1)s_2] = 0. \end{aligned}$$

Each factor in the parentheses represents a system of double contact conics, belonging to the pencil invariant under a dihedral G_{2n}, whose fixed point-line is a vertex and opposite side of the triangle of reference.

The system of conics K_c cut each other in $6n(3n-1)$ points, all but $9n^2$ of which coincide at the vertices of the reference triangle. These remaining points in general divide into a special set of $3n^2$ lying on the axes of reflexion and a general conjugate set lying on the system of lines $\Lambda_{-\omega}$. Exceptions are noted below.

If now we put $s_1 = 0$ in (16) and (17) respectively, we obtain the two equations

$$(18) \qquad (a^2-a+1)^3 s_3^2 + a^2(a-1)^2 s_2^3 = 0$$

and

$$(19) \qquad (c+1)^3 s_3^2 + c s_2^3 = 0.$$

Thence it appears that the system of lines Λ_a as well as the system of conics K_c cut s_1 in the same points as the pencil of invariant curves

$$(20) \qquad s_3^2 + \lambda s_2^3 = 0 \ *$$

$$(21) \qquad \lambda = a^2(a-1)^2/(a^2-a+1)^3 = c/(c+1)^3.$$

The correspondence between the curves of the pencil (20) and the sets of lines Λ_a is one-to-three and not one-to-six as a first glance at relation (21) suggests. For reciprocal values of a determine the same set of lines. Likewise the correspondence between the curves of the pencil and the systems of conics K_c is one-to-three in virtue of (21).

Now observing that when $s_1 = 0$, Δ reduces to $27 s_3^2 + 4 s_2^3 = 0$, we see that the curve (20) cuts s_1 in the same points as Δ when $\lambda = 4/27$. We have therefore the interesting result:

All sets of conjugate points on s_1, whether special or general, are the complete intersections of the curve with (a) *the systems of lines Λ_a,* (b) *the systems of conics K_c, and* (c) *the pencil of curves* (20).

The special sets are cut out by special curves as follows:

When	the curves Λ_a, K_c, (20) cut out
$\lambda = 0$,	(hyperosculation points) ^{2n};
$\lambda = \infty$,	(the $2n^2$ points)^3;
$\lambda = 4/27$,	(the $3n^2$ points)^2;

where the exponents indicate the multiplicity of intersection. General conjugate sets are cut out by single members of the pencil (20) but by triple sets of lines and conics because of the multiple correspondence. Some exceptions, however, occur in the case of special sets. When $\lambda = \infty$, there is a single set of lines, viz. $a = -\omega$, or $-\omega^2$, cutting out the $2n^2$ points, one line from each vertex of the reference triangle passing through each of the points. When $\lambda = 0$, there are two systems of lines cutting out the hyperosculation points for when $a = 0$ or ∞, the lines reduce to s_3^2, whereas when $a = 1$, the lines become the hyperosculation tangents repeated. Again, when $\lambda = 4/27$, we have from (21) $[(a+1)(a-2)(2a-1)]^2 = 0$, whence there

* It should be noted that this pencil is composite when $n \equiv 0$, mod 3.

are two sets of lines, namely, $a = -1$ and $a = 2$ (or $1/2$). We saw that $\Lambda_1 = \Delta$. We observe now that $\Lambda_2 - \Delta \equiv 9s_1^2 s_2^2$, hence Λ_2 cuts s_1 and s_2 in the same points as Δ. Or, *the system of lines Λ_2 intersect in pairs at the special set of $3n^2$ points both of s_1 and s_2.*

A single set of conics K_0 and K_{-1} respectively cut out the hyperosculation points and the $2n^2$ points. Each conic of the set K_{-1} contains $2n$ of the $2n^2$ points while each of the points is on three of the conics. But when $\lambda = 4/27$, $c = \frac{1}{2}, \frac{1}{2}, -4$ and there are two systems of conics. Two conics of the set $K_{1/2}$ cut s_1 at each of the $3n^2$ points, whereas one conic of the set K_{-4} touches s_1 at each of the points. Furthermore, we find $8K_{1/2} - \Delta \equiv s_1^2(s_2^2 - 2s_1 s_3) \equiv s_1^2(y^{2n}z^{2n} + z^{2n}x^{2n} + x^{2n}y^{2n})$, hence the remaining intersections of the system $K_{1/2}$ with the axes of reflexion fall on a curve of the type s_2 of double the order.

A second solution of the problem, involving the special sets symmetrically, is not without interest. We may regard each special set of points as composed of $6n^2$, each point counted 2, 3 or $2n$ times. Then recalling that these special sets of points are cut out by Δ, s_2^3 and s_3^2, and taking J for a parameter, we have the result:

All sets of conjugate points on s_1 are determined by the system of curves

$$J : J - 1 : 1 = -27s_3^2 : \Delta : 4s_2^3. \tag{22}$$ *

The construction of conjugate sets on s_2 is entirely analogous. We should naturally seek a pencil of curves of order $3n$. However, the systems of lines Λ_a and conics K_c are still available, for half of the intersections with s_2 fall at the multiple points, leaving in general $6n^2$ free intersections. We shall state the results at once:

All sets of conjugate points on s_2, both special and general, are cut out by (a) *the systems of lines Λ_a,* (b) *the systems of conics K_c,* (c) *the pencil of curves*

$$s_3 + \lambda s_1^3 = 0, \dagger \tag{23}$$

the parameters satisfying the relations

$$\lambda = a^2(a-1)^2/(a^2 - a + 1)^3 = c^2/(c+1)^3. \tag{24}$$

When	the several curves cut out
$\lambda = 0$,	(the three multiple points)$^{2n^2}$;
$\lambda = \infty$,	(the $2n^2$ points)^3;
$\lambda = 4/27$,	(the $3n^3$ points)^2.

* Cf. Dyck, *loc. cit.*, p. 514.

† The pencil is composite when n is a multiple of 3.

The correspondence between the curves of the pencil (23) on the one hand and the systems of lines and conics on the other is one-to-three in general but exceptions occur in the case of special sets. When $\lambda = 4/27$, there are but two systems of conics for then $c = 2,\ 2,\ -1/4$. The system K_2 pass two and two through the $3n^2$ points, each conic containing $2n$ of the points. The remaining intersections of these conics with axes of reflexion fall on a curve like s_1 but of twice the order, for

$$K_2 - \Delta \equiv s_2^2(s_1^2 - 2s_2) \equiv s_2^2(x^{2n} + y^{2n} + z^{2n}).$$

Each conic of the set $K_{-1/4}$ touches s_2 at n of the $3n^2$ points.

Again, *all sets of conjugate points on s_2 are determined by the pencils*

$$J : J - 1 : 1 = -27s_3^2 : \Delta : 4s_1^3 s_3, \tag{25}$$

where the points cut out by s_3 appear as extraneous in each set, the pencils being of order $6n$.

The inverse pencils of invariant curves $s_1^2 + \lambda s_2 = 0$ and $s_2^2 + \lambda s_1 s_3 = 0$ contain numerous members with intersecting geometrical properties as exemplified by the rather extensive discussion for the case $n = 3$.*

6. *Extension to Higher Spaces.* Many of the properties of the group and its invariants are susceptible of immediate generalization. Thus in space S_r of r dimensions we should be concerned with the monomial group that leaves invariant the variety

$$x_0^n + x_1^n + x_2^n + \cdots + x_r^n = 0. \tag{26}$$

The order of the group is $(r+1)!\,n^r$, comprising the elements

$$x_0' = x_0,\quad x_1' = \epsilon^{i_1} x_1, \cdots, x_r' = \epsilon^{i_r} x_r, \qquad i_1,\ i_2, \cdots, i_r = 1, \cdots, n, \tag{27}$$

combined with those of the group of order $(r+1)!$ which permutes the variables in all possible ways. The group is thus in $(n^r, 1)$ isomorphism with the permutation group $G_{(r+1)}!$ on $r+1$ letters,—a fact which supplies a key to its essential structure. The n^r elements (27) constitute an invariant Abelian subgroup which corresponds to the identity in the multiple isomorphism.

The associated binary form is now of order $r+1$:

$$t^{r+1} - s_1 t^r + s_2 t^{r-1} - \cdots + (-1)^r s_r t + (-1)^{r+1} s_{r+1} = 0, \tag{28}$$

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* Winger, *loc. cit.*, § 6. There is one curve of each system on the n^2 points, viz. $\lambda = -3$. This curve in each pencil is composite:

$$s_1^2 - 3s_2 \equiv (wx^n + w^2y^n + z^n)(w^2x^n + wy^n + z^n)$$

and the other is the inverse of this.

whose roots are $x_0^n, x_1^n, \cdots, x_r^n$. The complete system of invariants of the group comprises the r elementary symmetric functions of the roots of (28), $s_1, s_2, \cdots, s_r$, together with the product of the variables $x_0 x_1 \cdots x_r$,* and the difference product

$$\begin{aligned}\Pi \equiv (x_0^n - x_1^n)(x_0^n - x_2^n)(x_0^n - x_3^n) \cdots (x_0^n - x_r^n) \\ (x_1^n - x_2^n)(x_1^n - x_3^n) \cdots (x_1^n - x_r^n) \\ (x_2^n - x_3^n) \cdots (x_2^n - x_r^n) \\ \cdots \\ (x_{r-1}^n - x_r^n).\end{aligned}$$

The square of Π is the discriminant of the binary form, an identity which implies a syzygy among the fundamental forms. Every invariant as well as each coefficient of every covariant of the binary form is an invariant of the group. Indeed from each such invariant of the group can be derived a family of invariants by inserting a parameter in every term.

The inversion

$$x_i' = 1/x_i, \qquad i = 0, 1, \cdots, r \tag{29}$$

has the same effect on the form (28) as the binary inversion $t' = 1/t$. Thus the pairs of invariants s_i, s_{r-i+1} are interchanged. Accordingly we may construct invariants of the inversion by forming isobaric expressions such as

$$s_i s_{r-i+1} + \lambda s_{i+1} s_{r-i}, \tag{30}$$

where λ is a parameter. Such invariants of the inversion can also be constructed from the coefficients of every invariant of the binary form. In particular, if any covariant of the binary form (including the form itself) is of even order, the middle coefficient is an invariant of the inversion. Again, every invariant of the binary form is an invariant of the inversion. In fact, Π, the square root of the discriminant of the binary form, is an invariant of the inversion. Thus the interrelation between the covariant system of the binary form and the invariants of the group and the inversion, noted in the plane, carries on to higher dimensions.

The collineation group in S_r effects on each of the $r+1$ reference spaces S_{r-1} a collineation group of the type in question for the next lower space S_{r-1}. Sections of fundamental invariants in S_r (except $x_0 x_1 \cdots x_r$) by any reference space in S_{r-1} are invariants of the group in S_{r-1}: all are fundamental invariants in S_{r-1} except the section of the difference product Π, which is the difference product in S_{r-1} multiplied by the nth powers of the variables.

* We use the product of the variables instead of s_{r+1} which is the product of their n-th powers.

In S_3 the group is in $(n^3, 1)$ isomorphism with the octahedral G_{24}.* The elements of the Abelian invariant subgroup are given by (27) when $r = 3$. The binary form (28) now reduces to the quartic

$$t^4 - s_1 t^3 + s_2 t^2 - s_3 t + s_4. \tag{31}$$

The fundamental invariants are the surfaces

$$\begin{aligned}
\Pi_4 &: \; x_0 x_1 x_2 x_3 \\
s_1 &: \; x_0^n + x_1^n + x_2^n + x_3^n \\
s_2 &: \; x_0^n x_1^n + x_0^n x_2^n + x_0^n x_3^n + x_1^n x_2^n + x_1^n x_3^n + x_2^n x_3^n \\
s_3 &: \; x_0^n x_1^n x_2^n + x_0^n x_1^n x_3^n + x_0^n x_2^n x_3^n + x_1^n x_2^n x_3^n \\
\Pi_{6n} &: \; (x_0^n - x_1^n)(x_0^n - x_2^n)(x_0^n - x_3^n)(x_1^n - x_2^n)(x_1^n - x_3^n)(x_2^n - x_3^n),
\end{aligned}$$

of which obviously Π_4 is the tetrahedron of reference, while Π_{6n} represents a system of $6n$ planes, passing in sets of n through the edges of the tetrahedron.

The inversion $x_i' = 1/x_i$, $i = 0, 1, 2, 3$ interchanges the surfaces s_1 and s_3 but leaves s_2 unaltered. Other invariants of the inversion are: the system of planes Π_{6n}, the two invariants of the binary quartic, the middle term in the Hessian and the sextic covariant of the quartic. The intersections of s_2 with the surfaces s_1 and s_3 are a pair of space curves invariant under the group but interchanged by the inversion. The intersection of s_1 and s_3 however is a curve invariant alike under the group and the inversion. The reference planes cut the surface s_1 and s_2 respectively in sets of four curves of the types $x^n + y^n + z^n$ and $y^n z^n + z^n x^n + x^n y^n$. Each curve, of course, is invariant under the ternary group in its plane, while the four curves in either set are a conjugate set under the quaternary group.

In this section we have sketched in very rough outline the problem for the higher spaces. A systematic investigation of the group and its invariants for an arbitrary value of n even in S_3 would doubtless yield much of interest alike for geometry and group theory.

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* For a treatment of this group when $n = 6$, see Musselman, *loc. cit.*

Plane Involutions of Order Three or Four.

By F. R. SHARPE.

1. In "The problem of Plane Involutions of Order $t > 2$," * the following general method of constructing families of involutions was given. Let $|C| = a_1C_1 + a_2C_2 = 0$ be a pencil of curves of genus p. Construct a system of curves $|C| + |\overline{C}|$ which contains $|C|$ with a residual $|\overline{C}|$ and such that a curve C_3 of $|C| + |\overline{C}|$ meets a curve of $|C|$ in t variable points. The equations $y_1 = C_1\overline{C}$, $y_2 = C_2\overline{C}$, $y_3 = C_3$ define an involution of order t from which a family of involutions can be derived if the dimensions R of $|C| + |\overline{C}| > 2$. The cases in which $R = 2$ were not considered. It will be shown that they lead to new involutions, particularly if $t = 3$ or 4.

2. If two systems $|C|$ and $|\overline{C}|$ having i variable intersections, are of grade (number of variable intersections) $n, \bar{n}$, genus $p, \bar{p}$ and vitual dimension $\gamma, \bar{\gamma}$ then for $|C| + |\overline{C}|$, $N = n + \bar{n} + 2i$, $P = p + \bar{p} + i - 1$, $R = r + \bar{r} + i$.

The pencil $|C|$ is of vitual dimension $-p + 1 + n$ and grade n if $n \leq p$ of its simple basis points are not basis points of $|C| + |\overline{C}|$ and $i = t - n$. The dimension of $|C|$ must be 0. The dimension of $|C| + |\overline{C}|$ is therefore $2 = (-p + 1 + n) + 0 + (t - n)$. Hence $p = t - 1$. Moreover $P = p + \bar{p} + t - n - 1$, and $N = t$ when the y points in which C_3 meets C are included in the basis points. For a regular system $R = N - P + 1$ and the super abundance $s = R - (N - P + 1)$. Hence $s = y = P + 1 - t = p + \bar{p} - n$.

3. Let $t = 3$ so that $p = 2$. With the pencil $C_4 : A^2 12B$ we find

$\lvert C\rvert$	$\lvert\overline{C}\rvert$	$\lvert C\rvert + \lvert\overline{C}\rvert$
$C_4 : A^2 8B 3C$	$C_3 : A8B$	$C_7 : A^3 8B^2 3C 2D$
$C_4 : A^2 9B 3C$	$C_3 : 9B$	$C_7 : A^2 9B^2 3C 3D$
$C_4 : A^2 B 10C$	$C_4 : AB^2 10C$	$C_8 : A^3 B^3 10C^2 3D$
$C_4 : A^2 7B 3C 2D$	$C_6 : A^2 7B^2 3C$	$C_{10} : A^4 7B^3 3C^2 2D 4E$
$C_4 : A^2 9B 2C$	$C_7 : A^3 9B^2 2C$	$C_{11} : A^5 9B^3 2C^2 4D$
$C_4 : A^2 B 8C 2DE$	$C_7 : A^2 B^3 8C^2 2D$	$C_{11} : A^4 B^4 8C^3 2D^2 E 5F$
$C_4 : A^2 11BC$	$C_7 : A11B^2C$	$C_{11} : A^3 11B^3C^2 6D$
$C_4 : A^2 8B 2CDE$	$C_{10} : A^4 8B^3 2C^2 D$	$C_{14} : A^6 8B^4 2C^3 D^2 E 6F.$

* F. R. Sharpe, *American Journal of Mathematics*, Vol. 50 (1928), pp. 627-635.

Thus in the first case $n=1$, $p=1$, $P=4$, $y=2$ and similarly for the others.

With the pencil $C6 : 8A^2BB'CC'$ we find

$\|C\|$	$\|\overline{C}\|$	$\|C\|+\|\overline{C}\|$
$C6 : 2A^2 6B^2CC'D$	$C_1 : 2A$	$C_7 : 2A^3 6B^2CC'DE$
$C6 : A^2 7B^2CC'DD'$	$C_1 : AC$	$C_7 : A^3 7B^2C^2C'DD'2E$
$C6 : A^2 7B^2CC'D$	$C_3 : 7BCD$	$C_9 : A^2 7B^3C^2D^2C'2E$
$C6 : 8A^2CC'DD'$	$C_4 : 8AC^2C'DD'$	$C_{10} : 8A^3C^3C'^2D^2D'^2 4E$

With the pencil $C_7 : A^3 10B^2$

$C_7 : A^3 2B^2 8C^2$	$C_1 : 2B$	$C_8 : A^3 2B^3 8C^2 2D$
$C_7 : A^3 9B^2C^2$	$C_3 : 9B$	$C_{10} : A^3 9B^3C^2 3D$

With the pencil $C_9 : 8A^3 2B^2C$

$C_9 : 2A^3 6B^3 2C^2D$	$C_1 : 2A$	$C_{10} : 2A^4 6B^3 2C^2D2F$
$C_9 : A^3 7B^3C^2D^2E$	$C_3 : 7BCE$	$C_{12} : A^3 7B^4C^3D^2E^2 3F$
$C_9 : A^3 7B^3 2C^2$	$C_3 : 7B2C$	$C_{12} : A^3 7B^4 2C^3 2D$
$C_9 : 6A^3 2B^3 2C^2D$	$C_6 : 6A^2 2B2C^2D$	$C_{15} : 6A^5 2B^4 2C^4D^2 4E$

With $C_{13} : A^5 9B^4$

$C_{13} : A^5 9B^4$	$C_3 : 9B$	$C_{16} : A^5 9B^5 3C$

If $t=4$, $p=3$. With the pencil $C_4 : 16A$

$\|C\|$	$\|\overline{C}\|$	$\|C\|+\|\overline{C}\|$
$C_4 : A11B3C$	$C_4 : A^2 11B$	$C_8 : A^3 11B^2 3C4D$
$C_4 : A14B$	$C_5 : A^3 14B$	$C_9 : A^4 14B^2 5C$
$C_4 : AB14C$	$C_6 : A^4B^2 14C$	$C_{10} : A^5B^3 14C^2 6D$
$C_4 : 7A6B3C$	$C_6 : 7A^2 6B$	$C_{10} : 7A^3 6B^2 3C6D$
$C_4 : 6A9B$	$C_6 : 6A^2 9B$	$C_{10} : 6A^3 9B^2 6C$
$C_4 : 11A2B3C$	$C_7 : 11A^2 2B$	$C_{11} : 11A^2 2B^2 3C7D$
$C_4 : A8B5C2D$	$C_7 : A^3 8B^2 5C$	$C_{11} : A^4 8B^3 5C^2 2D7E$
$C_4 : 10A5B$	$C_7 : 10A^2 5B$	$C_{11} : 10A^3 5B^2 7C$

and similarly for the other pencils of genus 3.

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A Special Prepared System for two Quadratics in N Variables.

By J. WILLIAMSON.

Introduction. A complete system of concomitants for two quadratics in n variables is a system of concomitants, in terms of which every rational integral concomitant of the two quadratics may be expressed rationally and integrally. The existence of such a finite complete system is a particular case of Gordan's general theorem.* There are three possible types of complete systems,

$$G = G(a, r, u_1, u_2, \cdots, u_{n-2}, x),$$
$$H = H(a, r, x_1, x_2, \cdots, x_{n-1}, x_n),$$
$$K = K(a, r, \pi_1, \pi_2, \cdots, \pi_{n-1}, x),$$

where each contains coefficients a, r of the two quadratic ground forms, the u_i denote plane co-ordinates, x and x_i denote point co-ordinates, and π_i denote compound co-ordinates. From a geometrical point of view the K system is the most important, but also it is the largest and the most difficult to determine. This K system is known for the cases $n = 2$,† $n = 3$,‡ and $n = 4$.§ The number of irreducible concomitants for the cases $n = 2$, $n = 3$, and $n = 4$ are 6, 20 and 122 respectively. For the cases $n = 2$ and $n = 3$, the system is strictly irreducible and thus the complete system in these two cases is also the minimum system.¶

This system K is not the most general system, since such a system would contain other sets of variables ρ_i, σ_i, $\cdots$, all cogredient to π_i. But by Clebsch's theorem ‖ all concomitants involving such sets of variables ρ_i, σ_i, etc. can be deduced from the K system by polarization, if to the system K is added the actual concomitant of the field, the determinant of X (see § 1).

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* Grace and Young, *Algebra of Invariants*, Chap. 6.

† Grace and Young, *loc. cit.*, page 161.

‡ Grace and Young, *loc. cit.*, pp. 280-286.

§ H. W. Turnbull, "The Simultaneous System of Two Quadratic Quaternary Forms," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 18, parts 1 and 2 (1917), pp. 70-94. J. Williamson, "The Complete System of Two Quaternary Quadratics," *American Journal of Mathematics*, Vol. 51, No. 4 (October, 1929).

¶ Van de Waerden, "Over het concomitanten-system van twee en drie ternaire quadratische vormen," *Amsterdam Koninklijke Akademie, Verslagen wisen natuurkundige*, Vol. 32 (1923), pp. 138-144.

‖ Turnbull, "The Theory of Determinants, Matrices and Invariants," pp. 248 sq.

The G system has been determined * and in a joint paper by H. W. Turnbull and the author,† a complete system is found, when the co-ordinates π_i (or p_i) are decomposed into their components u or x.

In this paper we are interested in the K system and, though no complete system is determined in the general case, a distinct step is made in that direction. If every concomitant of the two quadratics may be expressed as a product of symbolic factors, where each symbol occurs an even number of times in each product and distinguishing marks on equivalent symbols may be neglected, the totality of such factors is said to form a prepared system for the two quadratics. We determine here a prepared system, in terms of which every concomitant, multiplied by a suitable invariant factor, may be expressed. This system is comparatively simple and consists of $2^n - 1$ factors. In obtaining a prepared system, in terms of which every concomitant, without being multiplied by an invariant factor, may be expressed new factors must be introduced, when $n \geqq 4$. The number of such new factors is 1, 8, and 52, for the cases $n = 4$, 5, and 6 respectively. It is hoped that these results will be published later.

1. *Co-ordinate Systems.* Let $(u) = (u^1 u^2 \cdots u^n)$ denote a set of n independent variables and suppose that (u) represents hyperplane co-ordinates in a space of $n - 1$ dimensions. If $(u_1) = (u)$, $(u_2), \cdots, (u_n)$ are n sets of cogredient variables, such that the sets (u_i) are linearly independent, we have an n-rowed square matrix

$$U = \| u_i^j \|$$

such that the determinant of U is different from zero; i. e.

$$D = | U | \neq 0. \tag{1}$$

A complete co-ordinate system for this space of $n - 1$ dimensions is given by

$$\pi_r = (u_1 u_2 \cdots u_r) \qquad (r = 1, 2, \cdots, n-1),$$

where π_r denotes the set of $n!/r!(n-r)!$ determinants formed from the first r columns of U. If

$$X = \| x_i^j \|$$

is the n-rowed square matrix, whose elements x_i^j are the co-factors of the elements u_{n+1-i}^{n+1-j} in D, then by Jacobi's Ratio Theorem

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* H. W. Turnbull, "The Irreducible Concomitants of Two Quadratics in n Variables," *The Transactions of the Cambridge Philosophical Society*, Vol. 21, No. 8 (July, 1909), pp. 197-240.

† H. W. Turnbull and J. Williamson, "Further Invariant Theory of Two Quadratics in n Variables," *Proceedings of the Royal Society of Edinburgh*, Vol. 50, Part I, No. 2 (1929-30), pp. 8-25.

$$\pi_r = (u_1 u_2 \cdots u_r) = D^{r+1-n}(x_1 x_2 \cdots x_{n-r}) = D^{r+1-n} p_{n-r},$$

where p_{n-r} is the set of $n!/r!(n-r)!$ determinants formed form the first $n-r$ columns of X. Hence we have the two dual co-ordinate systems: *

$$\begin{aligned}
(u) &= \pi_1 = D^{2-n} p_{n-1}, \\
(u_1 u_2) &= \pi_2 = D^{3-n} p_{n-2}, \\
&\cdots\cdots\cdots\cdots \\
(u_1 u_2 \cdots u_{n-2}) &= \pi_{n-2} = D^{-1} p_2, \\
(u_1 u_2 \cdots u_{n-1}) &= \pi_{n-1} = (x_1) = (x),
\end{aligned}$$

and finally

$$|X| = D^{n-1}.$$

2. *Notation.* Let

$$f = a_x^2 = b_x^2 = c_x^2 = \cdots, \quad g = r_x^2 = s_x^2 = t_x^2 = \cdots \tag{2}$$

be the symbolic forms of two quadratics in n variables, where $a_x = \sum_{i=1}^{n} a_i x_i$. Further let $A_i = B_i = C_i$ denote a matrix of n rows and i columns formed by i equivalent symbols of f, and $R_i = S_i = T_i$ a matrix of n rows and i columns formed by i equivalent symbols of g. We then say that A_i or R_i is of currency i. If, also, X_k denotes a matrix of n rows and k columns formed by k sets of variables cogredient with (x), and U_k a similar matrix formed by k sets of cogredient variables, each contragredient to the set (x), we may denote the ordinary determinantal bracket factor by

$$(A_i R_j U_k) \qquad (i+j+k=n,\ i \geqq 0,\ j \geqq 0,\ k \geqq 0).$$

We shall however require other types of bracket factors: I. Compound Inner Products; II. Generalised Outer Products; III. Generalised Compound Inner Products.

I. *Compound Inner Products.* The Compound inner product

$$(A_i R_j \mid X_{i+j}) \qquad (i+j \leqq n,\ i \geqq 0,\ j \geqq 0) \tag{3}$$

is a determinant of $i+j$ rows and columns and is equal to

$$|a_x b_y \cdots r_z s_w \cdots| = \Sigma \pm a_x b_y \cdots r_z s_w \cdots, \tag{4}$$

where $A_i = ab \cdots$, $R_j = rs \cdots$, $X_{i+j} = xy \cdots zw \cdots$. For example

$$(ab \mid xy) = \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = a_x b_y - b_x a_y.$$

* For a more detailed account of dual co-ordinate systems, see Turnbull, "Determinants, Matrices and Invariants," pp. 86-88.

The determinantal permutation indicated by $\Sigma \pm$ in (4) is often written

$$a_x \dot{b}_y \cdots \dot{r}_z \dot{s}_w \cdots \text{ or } (a \mid \dot{x})(b \mid \dot{y}) \cdots (r \mid \dot{z})(s \mid \dot{w}) \cdots,$$

where the dots mean that the letters beneath them must be permuted in every possible way, a change in sign being made with each transposition. If $i+j=n$ in (3),

$$(A_i R_j \mid X_n) = (A_i R_j) \mid X_n \mid,$$

and if $i+j=1$, this inner product reduces to the ordinary simple inner product $(a \mid x) = a_x$ or $(b \mid x) = b_x$. Since the inner product (3) is a determinant, it may be expanded by Laplacian developments in various ways. In fact,

$$(A_i R_j \mid X_i Y_j) = (\dot{A}_i \mid X_i)(\dot{R}_j \mid Y_j) = (A_i \mid \dot{X}_i)(R_j \mid \dot{Y}_j),$$

where again the dots indicate a series of $(i+j)!/i!\,j!$ terms obtained by interchanging the symbols of A_i with those of R_j in every possible way (or else those of X_i with those of Y_j). Once again each interchange is accompanied by a change of sign. For example,

$$\begin{aligned}(abc \mid xyz) &= (a\dot{b} \mid xy)(\dot{c} \mid z) = (ab \mid \dot{x}\dot{y})(c \mid \dot{z}),\\ &= (ab \mid xy)(c \mid z) - (ac \mid xy)(b \mid z) - (cb \mid xy)(a \mid z),\\ &= (ab \mid xy)(c \mid z) - (ab \mid xz)(c \mid y) - (ab \mid zy)(c \mid x).\end{aligned}$$

It should be noted, that the dot notation does not mean that the symbols appearing in the same bracket factor should be interchanged.* Moreover

$$(5) \qquad (A_i R_j \mid p_{i+j}) = D^{n-i-j-1}(A_i R_j \pi_{n-i-j}).$$

II. *Generalised Outer Products.* The generalised outer product

$$(6) \qquad (A_i R_j \mid X_k) \qquad (i+j \geqq n,\ k = i+j-n)$$

is defined as $(\dot{B}_{n-j} R_j)(\dot{C}_k \mid X_k)$ where $B_{n-j} C_k = A_i$ and the dots indicate a series of $i!/k!(n-j)!$ terms. By the elementary fundamental identities,

$$(\dot{B}_{n-j} R_j)(\dot{C}_k \mid X_k) = (A_i \dot{S}_{n-i})(\dot{T}_k X_k), \text{ where } T_k S_{n-i} = R_j.$$

The simplest type of such a factor is

$$(7) \qquad (A_i R_{n+1-i} x) = (\dot{A}_{i-1} R_{n+1-i}) a_x \qquad (i = 2, 3, \cdots, n-1),$$

where $A_i = A_{i-1} a$.

III. *Generalised Compound Inner Products.*

* Turnbull, *loc. cit.*, pp. 83 sq.

If for brevity in (7) we denote $(A_iR_{n+1-i}x)$ symbolically by i_x, we can form a third type of factor, a generalised compound inner product, which is defined in a manner analogous to the definition of a compound inner product. The generalised compound inner product $(ij\ldots k\,|\,xy\cdots z)$ is a determinant of m rows and columns, where m is the number of symbols $i, j, \cdots, k$. It is equal to

$$|\,i_xj_y\cdots k_z\,| = \dot{i}_x\dot{j}_y\cdots\dot{k}_z = (i\,|\,\dot{x})\,(j\,|\,\dot{y})\cdots(k\,|\,\dot{z}).$$

3. *Determination of a Prepared System.* By the fundamental theorem of invariants, every concomitant T of the two quadratics (2) can be represented as a sum of terms, where each term is a product of factors of the types $(d_1d_2\cdots d_n)$, $(d_1d_2\cdots d_i\pi_{n-i})$, and d_x, together with D, and each symbol d represents a symbol of f or g and must occur exactly twice in every product. But by (5)

$$D^{n-1}(d_1d_2\cdots d_n) = |\,X\,|\,(d_1d_2\cdots d_n) = (d_1d_2\cdots d_n\,|\,X), \tag{8}$$

and

$$D^{n-i-1}(d_1d_2\cdots d_i\pi_{n-i}) = (d_1d_2\cdots d_i\,|\,p_i). \tag{9}$$

Hence if M be any such term of T,

$$D^kM = K, \tag{10}$$

where K is a product of factors of the types on the right of equations (8) and (9) and k is the total number of variables x_i distinct from x, which occur in M. We now prove:

LEMMA I. *If K has the equivalent symbols $a, b, \cdots, c$, i in number, convolved together as A_i in the same factor, the complementary symbols $a, b, \cdots, c$ appearing in K may also be convolved together as A_i.*

Since every factor occurring in K is a compound inner product, we may write,

$$K = \Sigma(A_i\,|\,X_i)a_xb_y\cdots c_zN.$$

As $a, b, \cdots, c$ are equivalent symbols, by permuting them determinantally we get

$$i!\,K = \Sigma(A_i\,|\,X_i)\,(A_i\,|\,Y_i)N, \text{ where } Y_i = (xy\cdots z).$$

This proves the lemma. We apply the lemma successively for the cases $i = n$, $n-1$, etc. and finally have the result:

LEMMA II. *The concomitant K is zero, or else it can be expressed as a sum of terms, where each term is a product of factors of the two types*

$(A_i \mid Y_i)$, $(R_j \mid Y_j)$ and each symbol A_i, R_j occurs exactly twice in each product.

Further, if K has the factor $(A_n \mid X_n)$ or $(R_n \mid X_n)$, K has the invariant factor $(A_n \mid X_n)^2$ or $(R_n \mid X_n)^2$, or else is zero, since there are only n sets of variables (x). In either case we say that K is reducible.

Now two quadratics in n variables have the $n+1$ invariants

$$(A_i R_{n-i})^2 \qquad (i = 0, 1, \cdots, n), \tag{11}$$

which form a complete system of invariants,* and n quadratic covariants: the two original quadratics a_x^2, r_x^2 and the $n-2$

$$(A_i R_{n+1-i} x)^2 \qquad (i = 2, 3, \cdots, n-1) \tag{12}$$

defined by (7).† Let us denote the quadratic covariants symbolically by 1_x^2 and n_x^2 for a_x^2 and r_x^2 respectively, and by i_x^2 for $(A_i R_{n+1-i} x)^2$ $(i = 2, 3, \cdots, n-1)$. We require the following lemma:

LEMMA III. *The generalised inner product*

$$(i, i+1, i+2, \cdots, i+k \mid X_{k+1}) = \prod_{j=i}^{i+k-1} (A_j R_{n-j})\,(A_{i+k} R_{n+1-i} \mid X_{k+1})$$

together with additional terms.

For, if $s \geqq 0$ and $A_{i+k} = A_k A_i$,‡

$$\begin{aligned}
&(\dot{A}_i R_{n-i})\,(\dot{A}_k B_{t-s} U_{n-i+s-k}) \\
&= (B_{t-s} \dot{U}_s R_{n-i})\,(A_{i+k} \dot{U}_{n-i-k}) + \sum_{t=1}^{t-s} (\dot{B}_{t-s-t} \bar{U}_{s+t} R_{n-i})\,(A_{i+k} \dot{B}_t \bar{U}_{n-i-k-t}), \\
&= (B_{t-s} \dot{U}_s R_{n-i})\,(A_{i+k} \dot{U}_{n-i-k}) + \sum_{t=1}^{t-s} (B_{t-s-t} \dot{U}_{s+t} R_{n-i})\,(A_{i+k+t} \dot{U}_{n-i-k-t}), \\
&= (B_{t-s} \dot{A}_s R_{n-i})\,(\dot{A}_{i+k-s} U_{n-i-k-s}) + \sum_{t=1}^{t-s} (B_{t-s-t} \dot{A}_{s+t} R_{n-i})\,(\dot{A}_{i+k-s} U_{n-i-k+s}),
\end{aligned}$$

where $A_{i+k} B_t = A_{i+k+t}$. In all but the first term the currency of A_i has been increased at the expense of the currency of B_{t-s}, which is less than or equal to that of A_i. By (5) we may change the variables throughout and write

$$(\dot{A}_i R_{n-i})\,(\dot{A}_k B_{t-s} \mid X_{i+k-s}) \equiv (B_{t-s} \dot{A}_s R_{n-i})\,(\dot{A}_{i-k-s} \mid X_{i+k-s}) \tag{13}$$

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* Turnbull, *loc. cit.*, page 304.

† Turnbull and Williamson, "The Minimum System of Two Quadratics in n Variables," *Proceedings of the Royal Society of Edinburgh*, Vol. 45, Part 2 (1924-25), pp. 149-165.

‡ To avoid confusion a bar is sometimes used in place of a dot to denote determinantal permutations.

mod increased currency. We use $\equiv$ to denote "equal to except for terms, which may be neglected," and the words following *mod* to indicate why these terms may be neglected. We proceed to prove lemma III by induction, assuming its truth for the value k. Hence, if

$$L=(i, i+1, \cdots, i+k, i+k+1 \mid X_{k+2}),$$

by hypothesis

$$L \equiv \prod_{j=i}^{i+k-1} (A_j R_{n-j})\,(\dot{A}_{i-1} R_{n+1-i})\,(\overline{B}_{i+k} R_{n-i-k})\,(\dot{A}_{k+1}\overline{b} \mid X_{k+2}),$$

$$\equiv \prod_{j=i}^{i+k-1} (A_j R_{n-j})\,(\dot{A}_{i-1} R_{n+1-i})\,(\overline{B}_{i-1}\dot{A}_{k+1} R_{n-i-k})\,(\overline{B}_{k+2} \mid X_{k+2}) \text{ by (13)},$$

$$\equiv \prod_{j=i}^{i+k-1} (A_j R_{n-j})\,(\overline{B}_{i-1} R_{n+1-i})\,(A_{i+k} R_{n-i-k})\,(\overline{B}_{k+2} \mid X_{k+2})$$

mod increased currency of R_{n+1-i} by transferring A_{i-1} from the second to the third factor. Accordingly

$$L \equiv \prod_{j=i}^{i+k} (A_j R_{n-j})\,(\dot{B}_{i-1} R_{n+1-i})\,(\dot{B}_{k+2} \mid X_{k+2}).$$

In addition

$$(i, i+1 \mid xy)=(\dot{A}_{i-1} R_{n-i+1})\,(\overline{B}_i S_{n-i})\,(\dot{a}\overline{b} \mid xy),$$
$$=(A_i S_{n-i})\,(\overline{B}_{i-1} R_{n-i+1})\,(\overline{B}_2 \mid xy)$$

by putting $k=1$ in the previous work. Hence the proof by induction is complete except when $i=1$ or $i+k=n$. But, if $i=1$, the term $(A_{i-1} R_{n+1-i})$ disappears and by an otherwise similar proof

$$(14) \qquad (1, 2, \cdots, k \mid X_k) \equiv \prod_{i=1}^{k-1} (A_i R_{n-i})\,(A_k \mid X_k),$$

and similarily

$$(15) \qquad (n, n-1, \cdots, n-k+1 \mid X_k) = \prod_{i=1}^{k-1} (R_i A_{n-i})\,(R_k \mid X_k).$$

Though not necessary at this stage, it is important to notice for future applications that the terms neglected in lemma III have all had their currency raised.

It follows from lemma III that, if K be multiplied by the invariant factor $\prod_{i=1}^{k-1} (A_i R_{n-i})^2$ for every factor $(A_k \mid X_k)(A_k \mid Y_k)$ occurring in it and by $\prod_{i=1}^{k-1} (R_i A_{n-i})^2$ for every factor $(R_k \mid X_k)(R_k \mid Y_k)$ occurring in it, and if I denote this total invariant factor, IK can be expressed as a sum of terms, where each term is a product of factors of the types $(1, 2, \cdots, k \mid X_k)$ and $(n, n-1, \cdots, n-k+1 \mid X_k)$. What we have done is to replace each

symbol $a, b, \cdots, r, s, \cdots$ occurring in K by a symbol i occurring in IK. We can therefore convolve the variables back again exactly as they were originally convolved in K and have the final result:

The concomitant $IK = \Sigma N$, where N is a product of factors of the type $(i, j, \cdots, k \mid p_m)$ and each symbol $i, j, \cdots, k$ must occur exactly twice in each product.

But each $N = \Sigma I_i G_i$, where I_i is an invariant factor composed of powers of $D_j = (A_j R_{n-j})^2$, and G_i is a concomitant involving the symbols A_i, R_j convolved in pairs. Hence, if we determine the complete system for the G_i, we have determined a system in terms of which every concomitant of the two quadratics, if multiplied by an invariant factor composed of powers of D_i $(i = 1, 2, \cdots, n-1)$, can be expressed.

Order in which the concomitants G_i are considered. Let $\mu_{i,j}$ denote the number of pairs of symbols A_j occurring in G_i and $\nu_{i,j}$ the number of symbols R_j occurring in G_i. We consider G_1 before G_2, if G_1 is of less total degree in the coefficients of the two quadratics (2). But, if G_1 and G_2 are of the same degree in the coefficients of the two quadratics, we consider G_1 before G_2, if

$$\mu_{1,i+k} = \mu_{2,i+k},\ \nu_{1,i+k} = \nu_{2,i+k} \qquad (k = 1, 2, \cdots, n-i)$$

and

$$\mu_{1,i} > \mu_{2,i},\ \nu_{1,i} \not< \nu_{2,i}, \text{ or } \nu_{1,i} > \nu_{2,i},\ \mu_{1,i} \not< \mu_{2,i}.$$

In other words, G_1 is considered before G_2, if the currency of G_1 is greater than the currency of G_2.

That this is a legitimate procedure follows from the fact, that each G_i may be treated as a concomitant K and may therefore be expressed in terms of the symbols $i, j, \cdots, k$, in which the symbols A_t, R_m convolved in G_i will also be convolved. For a similar reason, if in G_i a new convolution of symbols A or R is made, the complementary symbols may also be convolved together. Finally, if A_n or R_n appears in G_i, A_n^2 or R_n^2 is a factor of G_i and G_i is said to be reducible.

As a direct consequence of the order in which the concomitants G_i are considered, we may use formulae (14), (15) and lemma III in determining the G_i, since in lemma III the neglected terms are all of greater currency.

Bracket Factors with equivalent symbols. If Θ denote a convolution of symbols $i, j, \cdots, k$ and Φ a similar convolution, while no symbol i occurs

in Θ equivalent to a symbol j occurring in Φ and no symbol in Θ is a symbol successive to one in Φ, we may write

$$(\Theta\Phi \mid p) = (\Theta \mid \dot{X})(\Phi \mid \dot{Y})$$

without disturbing the invariant factors D_i, which may have been introduced by (14) and (15). Hence in considering factors with equivalent symbols we need only consider factors of the one type

$$(i, i+1, \cdots, i+k, t, s, \cdots, m \mid X) \qquad (i \leqq t, s, m \leqq i+k).$$

If the symbols t, s, m are not successive integers, we may break the factor up again into a series of products of factors without disturbing the invariant factors D_i. Therefore all factors involving equivalent symbols may be reduced to the type

$$T = (i, i+1, \cdots, i+k, t, t+1, \cdots, t+s \mid X) \qquad (i \leqq t;\ i+k \geqq t+s),$$

where of course the variables p have been decomposed. Writing for i its value $(A_i R_{n+1-i})$ we see that T involves a convolution of

$$k+1+s+1 = k+s+2$$

equivalent symbols a. Since A_{i+k} is the symbol A of greatest currency appearing in T and since none of the symbols R have been disturbed, if $k+s+2 > i+k$, the currency of the resulting G_i has been increased. Hence, according to our scheme of order, G_i is reducible, if G_i contain the factor T, in which $s+2 > i$. We now consider the case in which $s+2 \leqq i$. If I denote the invariant factors, which appear in T by lemma III,

$$\begin{aligned} T &\equiv I(\dot{A}_{i-1} R_{n+1-i})(\bar{B}_{t-1} S_{n+1-t})(\dot{A}_{k+1} \bar{B}_{s+1} \mid X_{k+s+2}), \\ &\equiv I(\bar{B}_{s+1} \dot{A}_{i-2-s} R_{n+1-i})(\bar{B}_{t-1} S_{n+1-t})(\dot{A}_{k+s+2} \mid X_{k+s+2}) \end{aligned}$$

mod increased currency by (13), since $i-1 \geqq s+1$. But $n+1-i \geqq n+1-t$ and by transferring B_{s+1} from the first to the second factor we displace some of S_{n+1-t} and so increase the currency of R_{n+1-i}. This proof fails, if $i-1 < 1$, that is, if $i < 2$ or $i = 1$. But in this case $s+2 > i$, since s is not negative. Therefore in every case, if G_i contain a factor T involving equivalent symbols, G_i is reducible.

Since $i_x i_y' - i_x' i_y = (ii' \mid xy)$, which is reducible by the above, we may remove distinguishing marks on the equivalent symbols i, for, when i and i'

are interchanged, the invariant factors D_i will not be disturbed. We have now proved the theorem:

THEOREM I. *Every concomitant K of the two quadratics (2) is reducible or else, if multiplied by an invariant factor, composed of powers of $(A_iR_{n-i})^2$ $(i=1, 2, \cdots, n-1)$, is a sum of terms, where each term is a product of the symbolic factors i_x, $(ij \mid p_2)$, $(ijk \mid p_3), \cdots, (ijk\cdots m \mid p_{n-1})$, $(12\cdots n)$ where $i, j, k, m = 1, 2, \cdots, n$, and each symbol i occurs an even number of times in each product, while no equivalent symbols occur in the same factor. If the concomitant K is reducible, $K = IJ$, where I is an invariant factor composed of powers of $(A_n)^2$ and $(R_n)^2$ and J is not reducible.*

Since the variables are now convolved back again into the form p_j, we may replace the variables p by the variables π, thus removing the powers of D, with which we originally multiplied the concomitants. In fact, since the co-ordinates are homogeneous, it is immaterial which set of variables is used, as long as the variables are properly convolved. Henceforward we shall drop the variables and write (ij) for $(ij\pi_2)$, (ijk) for $(ijk\pi_3)$ etc.

4. *Identities.* If I_m, J_m, K_m etc. denote convolutions of m symbols $i, j, k, \cdots$ the following identities exist:

if $i+j+k=i+r+m$,

$$(16)\qquad \begin{cases} (I_i\dot{J}_jK_k)(I_i\dot{J}_rM_m) \equiv (I_iJ_{j+r}\dot{K}_{k-r})(I_i\dot{K}_rM_m) & (k>r), \\ \equiv (I_iJ_{j+r})(I_iK_kM_m) & (k=r), \\ \equiv 0 & (k<r). \end{cases}$$

For, if the variables are taken as p, in transferring J_r to the first factor we cannot displace any of the variables, since otherwise the second factor would be zero. But, if $i+j+k=i+r+m-1$,

$$(17)\qquad \begin{cases} (I_i\dot{J}_jK_k)(I_i\dot{J}_rM_m) \equiv (I_iJ_{j+r}\dot{K}_{k-r})(I_i\dot{K}_rM_m)+(I_iJ_{j+r}\dot{K}_{k-r+1})(I_i\dot{K}_{r-1}M \\ \qquad\qquad (k>r-1) \\ \equiv (I_iJ_{j+r})(I_iK_{r-1}M_m) \qquad (k=r-1) \\ \equiv 0 \qquad (k<r-1) \end{cases}$$

The two terms in this identity occur, because one of the variables in the first factor can be displaced. Other identities exist, when $i+j+k<i+r+m-1$,

but they disturb the proper convolution of the variables. These identities (16) and (17) are true whether the variables p or π are used.*

5. *Principle of Duality.* Let K be a concomitant $\Pi(ij\cdots k)$. Then, if $i', j', \cdots, k'$ are the symbols complementary to $i, j, \cdots, k$, that is, if $(ij\cdots k)(i'j'\cdots k')$ involves each of the symbols $1, 2, \cdots, n$ once and only once, $(ij\cdots k)$ and $(i'j'\cdots k')$ are said to be dual factors. With this notation, $\Pi(ij\cdots k)\Pi(i'j'\cdots k')$ involves all the symbols $1, 2, \cdots, n$ exactly m times if there are m factors in K. If m is even, all the symbols occur an even number of times and therefore $\Pi(i'j'\cdots k')$ is also a concomitant. But, if m is odd $(12\cdots n)\Pi(i'j'\cdots k')$ is a concomitant. Hence from a concomitant K we can form a dual concomitant by writing the dual factor of each factor in K and multiplying by the factor $(12\cdots n)$, if the number of factors in K is odd. If we consider identity (16) and write the dual of each factor in it, we get,

$$(TM_m\dot{J}_r)(T\dot{J}_jK_k)\equiv(T\dot{K}_rM_m)(TJ_{j+r}\dot{K}_{k-r}),$$

where $(TM_mJ_{j+r}K_kI_i)=(12\cdots n)$, and this new identity permutes exactly the same symbols as did the original one. Similarly there is an identity dual to (17) acting on the same symbols as (17). Hence we have the important lemma:

LEMMA IV. *Corresponding to any identity there is a dual identity operating on the same symbols.*

Since, if i and j are successive integers and ij is convolved an even number of times in a concomitant, an identity separating i and j cannot be used without disturbing the invariant factors, we must now prove a further lemma.

LEMMA V. *If K and K' are dual concomitants, the symbols i and j are convolved an even or an odd number of times in K' according as they are convolved an even or an odd number of times in K.*

Let the symbols i and j occur alone in m and r factors of K respectively, the symbols i, j occur together in t factors of K, and neither of the symbols i or j occur in q factors of K. Then the total number of factors in K is $m+r+t+q=s$. Since $t+m$ and $t+r$ are both even, $s+t\equiv q+m+t+r+t\equiv q \bmod 2$, or $q-t\equiv s \bmod 2$. *Hence*

$$t\equiv q \bmod 2, \text{ if } s \text{ is even},$$
$$\equiv q+1 \bmod 2, \text{ if } s \text{ is odd}.$$

* Turnbull, *loc. cit.*, pp. 92 sq.

But if s is even, q is the number of factors in K' containing both i and j, while, if s is odd, $q+1$ is the number of factors in K' containing both i and j, since, if s is odd, in determining K' from K we introduce the factor $(12\cdots n)$. Accordingly this proves our lemma.

6. *Incompleteness of the Prepared System.* We should like to determine a prepared system, in terms of which every concomitant can be expressed, without being multiplied by an invariant factor. To do this, we must see, if ever in convolving the variables to give the factors $(ij\cdots k)$ we have broken up a convolution of successive symbols. This is the case for $n>3$, and accordingly we must introduce other more complicated factor types. With the introduction of these new factor types, once we have found a complete system for the concomitants multiplied by invariant factors, we may remove the invariant factors from each concomitant and so obtain the actual complete system, provided that no identity is used, which disturbs a convolution of two successive symbols, if that convolution occurs an even number of times in a concomitant. These new factor types are of the nature

$$(\dot{\dot{i}}\dot{j}\cdots t)(\dot{k}\cdots\bar{s})(\bar{m}\cdots)(\bar{n}\cdots)\cdots(\bar{r}\cdots),$$

or in other words combinations of simple bracket factors, in which successive symbols are implicitly convolved. The dual of such a factor is obtained by taking the duals of the component bracket factors and permuting the same symbols. Accordingly, if one of these new factor types is counted as equivalent to q simple factors, where q is the number of component factors in the new factor type, the dual of a concomitant K containing such a factor can be written down as in § 4. Further the introduction of these new factor types does not vitiate the proof of lemma V, since any symbols implicitly convolved in K will also be implicitly convolved in K'. Hence from a list of irreducible concomitants we can immediately write down a list of the dual irreducible concomitants.

Since the concomitants f and g enter symmetrically into the discussion, the actual labour of determining the irreducible concomitants can be shortened still further. Two factors, concomitants or identities are said to be *similar*, if the first can be derived from the second by writing $n+1-i$ for every symbol i that occurs. From the symmetry mentioned above, it follows immediately that, if a concomitant is reducible, so is the concomitant similar to it.

7. *Applications of the General Theory to the Cases $n=2$ and 3. Complete System for Two Quadratics in 2 Variables.* In this case $n=2$ and we

require two variables, only one of which occurs explicitly. It is obvious that the prepared system consists of the three simple factors:

$$1_x = a_x, \quad 2_x = r_x, \quad (12) = (ar).$$

The prepared system then consists of six forms:

three invariants $(A_2)^2$, $(R_2)^2$, $(ar)^2$;
three covariants 1_x^2, 2_x^2, $(12)1_x2_x$.

In addition there is the actual concomitant of the field $D = (uv)$.*

Complete System for Two Quadratics in 3 Variables. In this case $n = 3$ and we require two co-ordinates x and $u = (xy)$. The prepared system consists of the seven simple factors:

$$1_x = a_x, \quad 3_x = r_x, \quad 2_x = (ARx) = (aR)b_x - (bR)a_x,$$
$$(12) = (aR)(Au), \quad (32) = (rA)(Ru), \quad (13) = (aru),$$
$$(123) = (aR)(Ar),$$

where A and R are written for A_2 and R_2 respectively. The complete system consists of 20 forms:

four invariants $(A_3)^2$, $(Ar)^2$, $(aR)^2$, $(R_3)^2$;
four covariants 1_x^2, 2_x^2, 3_x^2, $(123)1_x2_x3_x$;
four contravariants $(12)^2$, $(23)^2$, $(13)^2$, $(23)(31)(12)$;
eight mixed forms $(123)(23)1_x$, $(123)(12)3_x$, $(ij)i_xj_x$, $(123)(ij)(ik)i_x$,

where $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. In addition there is the actual concomitant of the field $D = (uvw)$. The actual concomitants are obtained from the above list by dropping the invariant factors, e. g. $(23)^2 = (rA)^2(Ru)^2$ gives the actual concomitant $(Ru)^2$.†

Determination of the Complete System. No new types of bracket factors are necessary, since the only chance of 12 or 23 being separated would be in the formation of the factor (123). But, as in (123) both 12, and 23 are convolved, no invariant factor would be disturbed. Accordingly the prepared system is that quoted above.

Covariants. If the factor (123) does not occur, the only possibilities are the squares of the factors i_x $(i = 1, 2, 3)$. If the factor (123) does occur, the only possibility is $(123)1_x2_x3_x$.

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* Weitzenböck, *Invariantentheorie*, Chap. 2, pp. 42, 43.
† Weitzenböck, *loc. cit.*, Chap. 2, page 61.

Contravariants. By the principle of duality we obtain the contravariants immediately from the list of covariants.

Mixed Concomitants. If the factor (123) does not occur, by analogy with binary forms the only possibilities are the three forms $(ij)i_x j_x$ $(i, j = 1, 2, 3)$. From these we obtain the dual forms $(123)(kj)(ki)k_x$ $(i, j, k = 1, 2, 3)$. If the factor (123) occurs, we may have the forms

$$(123)(23)1_x, \quad (123)(21)3_x, \quad (123)(13)2_x,$$

of which the last can be expressed in terms of the other two. It is obvious that three u factors cannot occur, since then we would have the concomitant

$$(123)(12)(23)(31)1_x 2_x 3_x$$

and this has the concomitant factor $(13)1_x 3_x$. Hence we have obtained the complete list of irreducible concomitants.

Similar methods have been employed to find the complete system for the case when $n = 4$.* When $n = 5$ and 6, an actual prepared system has been found by these methods, but the work is too long to be reproduced here.

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* J. Williamson, "The Complete System of Two Quarternary Quadratics," *American Journal of Mathematics*, Vol. 51, No. 4 (October, 1929). In this paper Theorem 1 is proved solely for the case $n = 4$.

On Algebraic Inversive Invariants.

By Frank Morley and Boyd C. Patterson.

Introduction. The invariant theory of the inversive group has been investigated by Kasner * most thoroughly. His mode of approach was by the study of "the theory of algebraic curves upon a proper quadric surface, with respect to those properties which are unaltered by the group of collineations transforming the quadric into itself." When the quadric is a sphere this group of collineations affects the points on the sphere in the same way as the group of inversions upon the sphere; and by an inversion of the sphere into a plane the geometry of the above mentioned theory of algebraic curves reduces to the inversive geometry of the plane.

It is the purpose of this paper to present new methods for calculating the algebraic invariants of curves under the inversive group. In particular, the invariants and covariants of the biquadratic are obtained. The method employed rests on the inversive significance of the equation of a curve given in circular coördinates. A bilinear equation represents a circle or a straight line (inversively equivalent to a circle) or two points, images in the plane considered. The biquadratic equation includes all circles which, with a circle, determine four points. Thus the curves otherwise called the conics, circular cubics, bicircular quartics, Cartesians, and Cassinians are all, inversively, biquadratics.

It is worthy of note that the methods presented may easily be generalized; and the results of the application of this method to the bilinear and bicubic forms are given.

In Section 4 an application of the results obtained for the general biquadratic is made to the Neuberg curve of a triangle. Sometimes called the Neuberg cubic (projectively speaking), from our point of view it is a biquadratic through infinity.

1. *Concomitants of the biquadratic.* We write the equation of the biquadratic

$$(1.1)\qquad (a_{22}x^2+2a_{21}x+a_{20})\bar{x}^2+2(a_{12}x^2+2a_{11}x+a_{10})\bar{x}+(a_{02}x^2+2a_{01}x+a_{00})=0$$

in circular coördinates with complex coefficients, $x=X-iY$ representing

* Edward Kasner, "The Invariant Theory of the Inversion Group: Geometry upon a Quadric Surface," *Transactions of the American Mathematical Society*, Vol. 1 (1900), p. 430.

a point of the complex plane and $\bar{x} = X - iY$ its reflexion in the axis of reals, i. e. its conjugate. An equation $f(x, \bar{x}) = 0$ represents a real curve only when it is self-conjugate. Hence we impose the condition that $\bar{a}_{ij} = a_{ji}$ from which it follows that a_{ii} is real. The coördinates may be modified by a real factor; thus we speak of the form itself as the curve.

In the sequel we shall frequently write the left member of (1.1) in the following matrix form:

(1.1′)

C_{12}	1	$2x$	x^2
1	a_{00}	a_{01}	a_{02}
$2\bar{x}$	a_{10}	a_{11}	a_{12}
$\bar{x}^2$	a_{20}	a_{21}	a_{22}.

By the notation $f \equiv C_{mn}$ we shall indicate that f is a form of degree m in the coefficients and of degree n in each of the variables x and $\bar{x}$. Equation (1.1) gives points on the biquadratic; a more general form of the equation is obtained by replacing $\bar{x}$ by $\bar{y}$. It is known from the general theory that an equation $f(x, \bar{y}) = 0$ does not represent a curve, since it is not self-conjugate, but rather associates with every point x of the plane certain points y. These points y are called image points of x with respect to the curve. In the case at hand we notice that for a given x we have a quadratic in $\bar{y}$ —hence two images y.

It can be shown with little difficulty that a $C_{12} = 0$ is transformed into a curve of the same type under transformations of the kind $y = \alpha x + \beta$ (a proportion or homology) and $z = 1/\bar{y}$ (an inversion in the base circle). Since the group of inversions consists of homologies and inversions in the unit circle, it follows that the type of the curve is left unaltered by the transformations of the inversive group

$$(1.2) \qquad y = (ax + b)/(cx + d), \qquad z = 1/\bar{y}.$$

The form C_{12} of the curve has therefore an inversive invariant significance. Furthermore the determinant of the coefficients is invariant under inversions. That is, if $C_{12} = 0$ is transformed into $C'_{12} = 0$ by a transformation (1.2) the expression

$$I_3 = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}$$

is, to within a constant factor, equal to the similar expression I'_3 of C'_{12}. In fact $I'_3 = (M\bar{M})^3 I_3$, $M = ad - bc$ being the modulus of the transformation. We note that the above matrix is an Hermitian matrix since $\bar{a}_{ij} = a_{ji}$.

Knowing I_3 to be an invariant of C_{12} we may calculate other invariants as follows. Writing down the invariant I_3 of the pencil of biquadratics

$$C_{12} + kP \qquad k, \text{ real}$$

and expanding in powers of k, we have invariants of the system appearing as coefficients of the various powers of k. If now P should be a degenerate biquadratic, say a point z taken four times,

$$P \equiv (x - z)^2(\bar{x} - \bar{z})^2 = 0,$$

we find the coefficients of k to be invariants of the system $C_{12} + kP$ if z is considered a constant or covariants of C_{12} if z is taken to be a variable. The invariant of the pencil $C_{12} + kP$ reduces to

$$kC_{22} + I_3.$$

The coefficient of k is a covariant of the original form of degree two in the coefficients a_{ij}—it is the first evectant * of C_{12}. Changing the variable from z to x, the matrix form of C_{22} is

	1	x	x^2
1	A_{22}	A_{21}	A_{20}
$\bar{x}$	A_{12}	A_{11}	A_{10}
$\bar{x}^2$	A_{02}	A_{01}	A_{00}

where A_{ij} is the minor of a_{ij} in I_3. We notice the binomial coefficients to be lacking in the C_{22} although they were present in C_{12}. If now we form the first evectant of the pencil $C_{22} + kP$ we find it to be precisely $I_3 C_{12}$. This reciprocal relation between the two forms C_{12} and C_{22} leads us to speak of C_{22} as the reciprocal of C_{12}.

The invariants of the system

$$C_{12} + kC_{22}$$

will be invariants of the form C_{12}. We obtain these invariants as coefficients of powers of k in the expansion of the following determinant which is the invariant I_3 of the pencil $C_{12} + kC_{22}$:

$$\begin{vmatrix} a_{00} + kA_{22} & 2a_{01} + kA_{21} & a_{02} + kA_{20} \\ 2a_{10} + kA_{12} & 4a_{11} + kA_{11} & 2a_{12} + kA_{10} \\ a_{20} + kA_{02} & 2a_{21} + kA_{01} & a_{22} + kA_{00} \end{vmatrix} = I_3^2k^3 + 2I_3I_2k^2 + I_4k + 4I_3.$$

* Cf. Salmon, *Modern Higher Algebra*, 2nd edition (1866), p. 105.

The values of these new invariants are:

$$I_2 = 2a^2{}_{11} + a_{00}a_{22} + a_{20}a_{02} - 2a_{10}a_{12} - 2a_{01}a_{21}.$$
$$I_4 = A^2{}_{11} + 8A_{00}A_{22} + 8A_{02}A_{20} - 4A_{10}A_{12} - 4A_{01}A_{21}.*$$

The covariants are obtained as coefficients of the powers of k in the development of the bordered determinant:

$$\begin{vmatrix} 0 & x^2 & -2x & 1 \\ \bar{x}^2 & a_{00}+kA_{22} & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ -2\bar{x} & \cdot\ \cdot\ \cdot & 4a_{11}+kA_{11} & \cdot\ \cdot\ \cdot \\ 1 & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & a_{22}+kA_{00} \end{vmatrix}$$
$$= -I_3C_{12}k^2 - (I_2C_{12} - 2C_{32})k - 4C_{22}.$$

The covariants C_{12} and C_{22} are the original form and its reciprocal. The third, a C_{32}, is as follows:

C_{32}	1	$2x$	x^2
1	α_{00}	α_{01}	α_{02}
$2\bar{x}$	α_{10}	α_{11}	α_{12}
$\bar{x}^2$	α_{20}	α_{21}	α_{22}

where the coefficients α_{ij} have the following values:

$$\alpha_{22} = a_{22}(a_{02}a_{20} - a^2{}_{11}) + 2a_{11}a_{21}a_{12} - a^2{}_{12}a_{20} - a^2{}_{21}a_{02}.$$
$$\alpha_{12} = a_{01}(a_{12}a_{21} - a_{11}a_{22}) + a_{02}(a_{10}a_{22} - a_{11}a_{21}) - a_{12}(a_{10}a_{12} - a^2{}_{11}).$$
$$\alpha_{02} = a_{00}(a_{02}a_{22} - a^2{}_{12}) - a^2{}_{01}a_{22} + 2a_{01}a_{11}a_{12} - a_{02}a^2{}_{11}.$$
$$2\alpha_{11} = a_{01}(a_{12}a_{20} - a_{11}a_{21}) + a_{02}(a_{10}a_{21} - a_{11}a_{20}) - 2a_{11}(a_{10}a_{12} - a^2{}_{11}) - a_{00}(a_{11}a_{22} - a_{12}a_{21}) + a_{01}(a_{10}a_{22} - a_{21}a_{11}).$$
$$\alpha_{01} = a_{00}(a_{02}a_{21} - a_{12}a_{11}) - a_{01}(a_{21}a_{01} - a^2{}_{11}) + a_{10}(a_{01}a_{12} - a_{11}a_{02}).$$
$$\alpha_{00} = a_{00}(a_{02}a_{20} - a^2{}_{11}) - a^2{}_{01}a_{20} + 2a_{01}a_{10}a_{11} - a^2{}_{10}a_{02}.$$
$$\bar{\alpha}_{ij} = \alpha_{ji}.$$

It is known that a biquadratic curve is its own inverse with respect to any one of its four Jacobian or "director" circles. Inversion with respect to an intersection of any two of these circles (which are orthogonal) sends the curve into a biquadratic symmetrically placed with respect to two perpendicular lines. Taking these lines as coördinate axes, the equation of the biquadratic assumes the normal form:

$$C_{12} \equiv \kappa(x^2\bar{x}^2 + 1) + \lambda(x^2 + \bar{x}^2) + 4\mu x\bar{x}.$$

The normal forms of the other covariants and invariants are:

$$C_{22} \equiv \kappa\mu(x^2\bar{x}^2 + 1) - \lambda\mu(x^2 + \bar{x}^2) + (\kappa^2 - \lambda^2)x\bar{x},$$
$$C_{32} \equiv \kappa(\lambda^2 - \mu^2)(x^2\bar{x}^2 + 1) + \lambda(\kappa^2 - \mu^2)(x^2 + \bar{x}^2) - 2\mu(\kappa^2 + \lambda^2 - 2\mu^2)x\bar{x},$$
$$I_2 = \kappa^2 + \lambda^2 + 2\mu^2, \quad I_3 = \mu(\kappa^2 - \lambda^2), \quad I_4 = (\kappa^2 - \lambda^2)^2 + 8\mu^2(\kappa^2 + \lambda^2).$$

---

* Kasner, *loc. cit.*, p. 479, gives the invariants I, J, K. In terms of the above, $I = -I_2$, $J = 2I_2$, $4K = I_3{}^2 - I_4$.

The above normal forms rule out the conics as special biquadratics. They suffice, however, to calculate important relations between the invariants.

Writing C_{22} with binomial coefficients as in (1.1′) we have the following relations between the invariants J of this form and the invariants I of C_{12}:

$$8J_2 = I_4, \qquad 4J_3 = I_3^2, \qquad 2J_4 = I_3^2 I_2.$$

Repeating the process on the reciprocal of C_{22} and indicating by K the invariants of this form, we have

(1.3) $\quad 16K_2 = 2J_4 = I_3^2 I_2, \quad 64K_3 = 16J_3^2 = I_3^4, \quad 256K_4 = 128J_3^2 J_2 = I_3^4 I_4.$

Foci. The image equation of the general *bi-n-ic* associates with any point x of the plane n image points y. By definition a focus of the curve is a point x which has two coincident images. Writing the biquadratic C_{12} in the form

$$(a_{22}x^2 + 2a_{21}x + a_{20})\bar{y}^2 + 2(a_{12}x^2 + 2a_{11}x + a_{10})\bar{y} + (a_{02}x^2 + 2a_{01}x + a_{00}) = 0$$

and requiring the discriminant of this quadratic in $\bar{y}$ to vanish makes evident the fact that $C_{12} = 0$ has four foci. The foci of the normal form are given by the quartic

$$Q \equiv \kappa\lambda(x^4 + 1) + (\kappa^2 + \lambda^2 - 4\mu^2)x^2 = 0.$$

Writing the biquadratic as a quadratic in x and requiring its discriminant to vanish gives four points y as foci which are identical to the four roots of Q. The projective invariants and covariants of Q are:

$g_2 = \kappa^2\lambda^2 + 3A^2, \qquad g_3 = A(\kappa^2\lambda^2 - A^2), \qquad$ where $6A = \kappa^2 + \lambda^2 - 4\mu^2$;

$\Delta_{12} \equiv g_2^3 - 27g_3^2$

$= \{(1/4)\kappa\lambda(\kappa + \lambda + 2\mu)(\kappa + \lambda - 2\mu)(\kappa - \lambda + 2\mu)(-\kappa + \lambda + 2\mu)\}^2,$

$H \equiv \kappa\lambda A(x^4 + 1) + (\kappa^2\lambda^2 - 3A^2)x^2, \qquad S \equiv \kappa\lambda(\kappa^2\lambda^2 - 9A^2)(x^5 - x),$

where the subscripts of Δ indicate its degree in the coefficients of C_{12}; H is the Hessian and S the sextic covariant. Foci invert into foci, hence we expect the above invariants of Q to be expressible in terms of the inversive invariants of C_{12}. In fact we have

$12g_2 = 4I_2^2 - 3I_4, \qquad 216g_3 = 8I_2^3 + 54I_3^2 - 9I_2I_4,$

$\Delta_{12} = (1/64)(I_2^2I_4^2 - I_4^3 - 108I_3^4 - 32I_2^3I_3^2 + 36I_2I_3^2I_4).$

The foci of C_{22} are given by

$Q' \equiv -\kappa\lambda\mu^2(x^4 + 1) - \{(1/4)(\kappa^2 - \lambda^2)^2 - \mu^2(\kappa^2 + \lambda^2)\}x^2 = 0.$

Between Q, Q', and H there exists the relation

$$Q' = -(I_2/6)Q + H.$$

Q' then is one of the pencil of quartics $mQ + nH$ and the invariants of Q' may be expressed in terms of the invariants of Q.* In fact

* Salmon, *loc. cit.*, p. 177.

$$
\begin{aligned}
&36g'_2 = I_2^2 g_2 - 18 I_2 g_3 + 3g_2^2, \\
&216g'_3 = -I_2^3 g_3 + I_2^2 g_2^2 - 9 I_2 g_2 g_3 + 54 g_3^2 - g_2^3, \\
&256\Delta'_{24} = I_3^4 \Delta_{12}, \qquad -16S' = I_3^2 S.
\end{aligned} \tag{1.4}
$$

In addition we have the identity

$$4H^3 - g_2 H Q^2 + g_3 Q^3 + S^2 \equiv 0,$$

and the invariants g_2 and g_3 expressed in terms of g'_2 and g'_3:

$$
\begin{aligned}
&3^2 I_3^4 g_2 = I_4^2 g'_2 - 2^4 \cdot 3^2 I_4 g'_3 + 2^6 \cdot 3 g'^2_2 \\
&3^3 I_3^6 g_3 = -I_4^3 g'_3 + 8 I_4^2 g'^2_2 - 8^2 \cdot 3^2 I_4 g'_2 g'_3 + 8^3 \cdot 2 \cdot 3^3 g'^2_3 - 8^3 g'^3_2.
\end{aligned}
$$

Two Biquadratics. We close this section with a word concerning the simultaneous invariants of two biquadratics C_{12} and C'_{12} with coefficients a_{ij} and b_{ij} respectively. Forming the pencil

$$C_{12} + kC'_{12}$$

and calculating the invariants corresponding to I_2, I_3, and I_4 of C_{12}, we obtain other invariants involving the coefficients of both forms. For future reference we give here the bilinear or apolar invariant of the two forms:

$$
\begin{aligned}
I_{11} = {} & 4a_{11}b_{11} + a_{00}b_{22} + a_{22}b_{00} + a_{20}b_{02} + a_{02}b_{20} \\
& - 2(a_{10}b_{12} + a_{12}b_{10} + a_{01}b_{21} + a_{21}b_{01}).
\end{aligned}
$$

2. *Invariants of circles.* Applying the method of the preceding section to the bilinear forms

$$
\begin{aligned}
C_{11} &\equiv \alpha_{11}x\bar{x} + \alpha_{01}x + \alpha_{10}\bar{x} + \alpha_{00} \\
C'_{11} &\equiv \beta_{11}x\bar{x} + \beta_{01}x + \beta_{10}\bar{x} + \beta_{00}
\end{aligned}
$$

gives us the invariant

$$I = \alpha_{00}\alpha_{11} - \alpha_{10}\alpha_{01}$$

of the single form, and

$$J = \alpha_{11}\beta_{00} - \alpha_{01}\beta_{10} - \alpha_{10}\beta_{01} + \alpha_{00}\beta_{11},$$

the simultaneous invariant of the two forms.

Geometrically interpreted, $C_{11} = 0$ is a circle which is a point when $I = 0$. The vanishing of J is the condition that the two circles be orthogonal.

If the four circles

$$
\begin{aligned}
f_1 &\equiv \alpha_{11}x\bar{x} + \alpha_{01}x + \alpha_{10}\bar{x} + \alpha_{00}, & f_2 &\equiv \beta_{11}x\bar{x} + \beta_{01}x + \beta_{10}\bar{x} + \beta_{00}, \\
f_3 &\equiv \gamma_{11}x\bar{x} + \gamma_{01}x + \gamma_{10}\bar{x} + \gamma_{00}, & f_4 &\equiv \delta_{11}x\bar{x} + \delta_{01}x + \delta_{10}\bar{x} + \delta_{00},
\end{aligned}
$$

have a common orthogonal circle

$$f \equiv a_{11}x\bar{x} + a_{01}x + a_{10}\bar{x} + a_{00}$$

then

$$J(f_k f) = 0 \qquad (k = 1, 2, 3, 4).$$

Eliminating a_{ij} from these four equations gives us the invariant condition that f_k have a common orthogonal circle:

$$K \equiv \begin{vmatrix} \alpha_{11} & \alpha_{01} & \alpha_{10} & \alpha_{00} \\ \beta_{11} & \beta_{01} & \beta_{10} & \beta_{00} \\ \gamma_{11} & \gamma_{01} & \gamma_{10} & \gamma_{00} \\ \delta_{11} & \delta_{01} & \delta_{10} & \delta_{00} \end{vmatrix} = 0.$$

Two circles $C_{11} = 0$ and $C'_{11} = 0$ have one absolute invariant,

$$J^2/II' = 4\cos^2\theta \text{ or } 4\cosh^2\delta,$$

where θ is the angle when the circles intersect; δ is the hyperbolic distance (measured from the common inverse points) when they do not. Hence two circles are tangent when

$$\begin{vmatrix} J & 2I \\ 2I' & J \end{vmatrix} = 0.$$

3. *Interpretations of the invariants of* C_{12}. Before entering upon the subject of this section we define several circles * intimately connected with the biquadratic C_{12}. The equations for the most part will be given in the normal form.

(α) The conjugate point x of a point z with respect to C_{12} is the harmonic conjugate of z with respect to the image points of z. It is given by the equation

$$(\kappa\bar{z}^2 + \lambda)xz + 2\mu\bar{z}(x+z) + (\kappa + \lambda\bar{z}^2) = 0.$$

(β) The bipolar circle C' of a point z with respect to C_{12} is the locus of double points of the one-to-one correspondence

$$(\kappa\bar{y}\bar{z} + \lambda)xz + \mu(\bar{y}+\bar{z})(x+z) + (\kappa + \lambda\bar{y}\bar{z})$$

determined by polarizing the image equation of C_{12} with respect to x and $\bar{y}$ simultaneously. Its equation is

$$C' \equiv (\kappa z\bar{z} + \mu)x\bar{x} + (\lambda z + \mu\bar{z})x + (\lambda\bar{z} + \mu z)\bar{x} + (\kappa + \mu z\bar{z}) = 0.$$

We note that the conjugate point of z is the inverse of z with respect to C'.

(γ) The bipolar circle C_1 of a circle C with respect to C_{12} is the locus of points whose bipolar circles are orthogonal to C. If the circle C has the equation

$$C \equiv \alpha_{11}x\bar{x} + \alpha_{01}x + \alpha_{10}\bar{x} + \alpha_{00} = 0$$

the equation of its bipolar circle is

$$C_1 \equiv (\alpha_{11}\mu + \alpha_{00}\kappa)x\bar{x} - (\alpha_{01}\mu + \alpha_{10}\lambda)x - (\alpha_{10}\mu + \alpha_{01}\lambda)\bar{x} + (\alpha_{11}\kappa + \alpha_{00}\mu) = 0.$$

* See Kasner, *loc. cit.*, pp. 482, 485 for definitions in quaternary form. Properly for circle we should write bilinear curve.

(δ) The anti-bipolar * circle C_2 of a circle C with respect to C_{12} is the circle orthogonal to the bipolar circles of all points on C. The bipolar circle of the anti-bipolar circle is the original circle. Hence the equation of the anti-bipolar circle of C is

$$C_2 \equiv (\lambda^2 - \mu^2)(\alpha_{00}\kappa - \alpha_{11}\mu)x\bar{x} + (\kappa^2 - \mu^2)(\alpha_{01}\mu - \alpha_{10}\lambda)x + (\kappa^2 - \mu^2)(\alpha_{10}\mu - \alpha_{01}\lambda)\bar{x} + (\lambda^2 - \mu^2)(\alpha_{11}\kappa - \alpha_{00}\mu) = 0.$$

(ϵ) When C reduces to a point z we have the anti-bipolar circle C'' of z with respect to C_{12}. Its equation is

$$C'' \equiv (\lambda^2 - \mu^2)(\kappa z\bar{z} - \mu)x\bar{x} + (\kappa^2 - \mu^2)(\lambda z - \mu\bar{z})x + (\kappa^2 - \mu^2)(\lambda\bar{z} - \mu z)\bar{x} + (\lambda^2 - \mu^2)(\kappa - \mu z\bar{z}) = 0.$$

A geometrical interpretation of the covariants is obtained by the application of the results of Section 2 to the bipolar circles P_{12} and P_{22} of a point with respect to C_{12} and C_{22} respectively. Since $I(P_{22}) = (I_3/4)C_{12}$, $C_{12} = 0$ is the locus of points whose bipolar circles with respect to C_{22} are degenerate. Also $C_{12} = 0$ is the original biquadratic and the bipolar circle with respect to C_{12} of each point on it passes through that point. $I(P_{12}) = C_{22}$, hence $C_{22} = 0$ is the locus of points whose bipolar circles with respect to C_{12} are degenerate. Finally, $C_{32} = 0$ is the locus of a point whose anti-bipolar circle with respect to C_{12} passes through it.

Calculating the bilinear or apolar † invariant of the biquadratics C_{12}, C_{22}, and C_{32} gives the following facts:

$$I_{11}(C_{12}C_{12}) = 2I_2, \qquad I_{11}(C_{22}C_{22}) = I_4/4,$$
$$I_{11}(C_{12}C_{22}) = 3I_3, \qquad I_{11}(C_{12}C_{32}) = I_2^2 - I_4.$$

Hence $I_2 = 0$ is the condition that the biquadratic is self-apolar. $I_3 = 0$ is the condition that the biquadratic is apolar to its reciprocal curve C_{22}. And when $I_4 = 0$ the covariant curve C_{22} is self-apolar. We give here another interpretation of the vanishing of I_3.

Writing $C_{12} = 0$ as an image equation and polarizing with respect to x and $\bar{y}$ defines a one-to-one correspondence between two points x and y:

$$[a_{22}xx_1 + a_{21}(x + x_1) + a_{20}]\bar{y}\bar{y}_1 + [a_{12}xx_1 + a_{11}(x + x_1) + a_{10}](\bar{y} + \bar{y}_1) + [a_{02}xx_1 + a_{01}(x + x_1) + a_{00}] = 0.$$

Now the condition that there be a neutral pair, i. e., a pair of points x and x_1 for which y is arbitrary, is that the three equations

$$a_{i2}xx_1 + a_{i1}(x + x_1) + a_{i0} = 0 \qquad (i = 0, 1, 2)$$

have a solution, that is $I_3 = 0$.

---

* J. L. Coolidge, *Treatise on the Circle and the Sphere*, Oxford (1916), p. 211, calls this the auto-polar circle.

† Kasner, *loc. cit.*, p. 471.

$I_3 = 0$ is the condition that C_{12} has a neutral pair. Neutral pairs in the above sense are characteristic of Cassinians and their inverses.

We have defined a focus as a point whose images with respect to C_{12} are coincident. A biquadratic will have a double point if two foci coincide and the double point then absorbs the foci. Three coincident foci give the biquadratic a cusp. From the invariants of Q (Section 1) we have:

$C_{12} = 0$ is a nodal biquadratic when $\Delta_{12} = 0$. A nodal biquadratic is a central conic or the inverse of a central conic. The conic is an ellipse if the node is isolated, an hyperbola if the node is an ordinary double point.

$C_{12} = 0$ is a cuspidal biquadratic when $4H^3 + S^2 \equiv 0$, for $g_2 = 0$ and $g_3 = 0$. A cuspidal biquadratic is a parabola or the inverse of a parabola.

$C_{12} = 0$ has two nodes and degenerates into a pair of circles when $S \equiv 0$. For then Q has two pairs of double roots and H is, to a constant multiple, the same as Q; hence the Jacobian S of H and Q vanishes identically.

The relations (1. 3) and (1. 4) give us corresponding results concerning the reciprocal curve $C_{22} = 0$ and make the reciprocity of C_{12} and C_{22} even more striking.

4. *The Neuberg curve of a triangle.** Given three points a_1, a_2, a_3, their Neuberg curve is defined to be the locus of a point a_4 such that, if λ_{ij} is the square of the distance between a_i and a_j,

$$\Delta' \equiv \begin{vmatrix} \lambda_{23}\lambda_{14} & \lambda_{23} + \lambda_{14} & 1 \\ \lambda_{31}\lambda_{24} & \lambda_{31} + \lambda_{24} & 1 \\ \lambda_{12}\lambda_{34} & \lambda_{12} + \lambda_{34} & 1 \end{vmatrix} = 0.$$

The Neuberg curve has the property that the isogonal conjugate with respect to the triangle $a_1a_2a_3$ of each point on the curve also lies on the curve and their join is parallel to the Euler line of the triangle. This definition of the curve enables us to derive its equation in circular coördinates. For we can take the vertices of the triangle on the unit circle, say, t_1, t_2, t_3, and choose the axes so that the axis of imaginaries goes through the orthocenter. With this choice the circumcenter of the triangle is the origin and the orthocenter $s_1 = t_1 + t_2 + t_3$ is a pure imaginary, say $s_1 = 2\rho i$ where ρ is real and positive. The transformation of isogonal conjugates for this triangle is

$$x + y + s_3\bar{x}\bar{y} = s_1, \qquad (s_3 = t_1t_2t_3)$$

and the Euler line is the axis of imaginaries which has a clinant (the square of its direction) of -1.

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* For bibliography, see B. H. Brown, "The 21-point Cubic," *American Mathematical Monthly*, Vol. 32 (1925), p. 110.

If the join of a pair of isogonal conjugates x, y is to be parallel to the Euler line we must have

$$(x-y)/(\bar{x}-\bar{y})=-1.$$

The locus of isogonal conjugates x, y satisfying

$$x+y+s_3\bar{x}\bar{y}=2\rho i, \qquad \bar{x}+\bar{y}+xy/s_3=-2\rho i, \qquad x+\bar{x}=y+\bar{y}$$

is the Neuberg curve of the triangle $t_1t_2t_3$. Eliminating 1, y, $\bar{y}$, from these three equations, we find the locus to be

$$(4.1) \qquad x^2\bar{x}+x\bar{x}^2+x^2/s_3+s_3\bar{x}^2-2(1+\rho i/s_3)x-2(1-\rho i s_3)\bar{x}=0.$$

We notice that the Neuberg curve is a biquadratic through infinity. (That this equation is the same as $\Delta'=0$ may be shown by direct substitution of t_i for a_i and of x for a_4).

A third characterization of the Neuberg curve is: The locus of points x whose image triangle with respect to the triangle $t_1t_2t_3$ is perspective to $t_1t_2t_3$. The equation of the curve in circular coördinates may also be derived from this definition.

It has been shown that the Neuberg curve is a symmetrical relation on five points,* and the property mentioned in the preceding paragraph tells what it is. Take any one of the five points, say α_4, and its three inverse points with respect to the three circles determined by $\alpha_2\alpha_3\alpha_5$, $\alpha_3\alpha_1\alpha_5$, and $\alpha_1\alpha_2\alpha_5$. Call these inverse points β_1, β_2, β_3, respectively. The distinctive property of the five points $\alpha_1, \cdots, \alpha_5$ is that the circles $\alpha_1\beta_1\alpha_5$, $\alpha_2\beta_2\alpha_5$, $\alpha_3\beta_3\alpha_5$, have in addition to α_5 a second common point, i. e. they are coaxal.

The Neuberg curve therefore is the locus of a point α_4 having the above relation with respect to four special points. The symmetry of $\Delta'=0$ in its inverted form shows that what is true inversively of one of the points is true of the others. Before we attempt to characterize these points inversively, i. e. by means of inversive invariants and covariants, we put them into a more convenient position by inverting with respect to one of them, say α_5. The four points now become the vertices of a triangle, which we call t_1, t_2, t_3, and the infinite point of the plane; and the equation of the curve is (4.1) which we shall indicate hereafter by C_{12}.

Applying the results of Section 1 we have the covariants

C_{12}	1	$2x$	x^2
1	0	$-(1+\rho i/s_3)$	$1/s_3$
$2\bar{x}$	$-(1-\rho i s_3)$	0	1/2
$\bar{x}^2$	s_3	1/2	0

* Frank Morley, "Note on Neuberg's Cubic Curve," *American Mathematical Monthly*, Vol. 32 (1925), p. 407.

C_{22}	1	x	x^2
1	$-(1+2\rho\sigma+\rho^2)$	$(1/s_3-\rho i)$	$-(1+\rho i/s_3)/2$
$\bar{x}$	$(s_3+\rho i)$	-1	$-1/2s_3$
$\bar{x}^2$	$-(1-\rho i s_3)/2$	$-s_3/2$	$-1/4$

C_{32}	1	$2x$	x^2
1	$2\gamma(\rho^2-1)$	$\rho\gamma(\rho/s_3-i)$	0
$2\bar{x}$	$\rho\gamma(\rho s_3+i)$	$-\gamma/2$	$-\rho i\gamma/2$
$\bar{x}^2$	0	$\rho i\gamma/2$	$-\gamma/2$

where s_3 is a complex number of absolute value unity and has been written, when convenient, $\cos\phi + i\sin\phi = \gamma + i\sigma$; $\phi = \text{amp}\, s_3$ and hence

$$2i\sigma = s_3 - 1/s_3, \qquad 2\gamma = s_3 + 1/s_3.$$

The invariants of the Neuberg curve are

$$I_2 = 3 + 2\rho\sigma, \qquad I_3 = -\gamma, \qquad I_4 = 9 + 12\rho\sigma + 4\rho^2.$$

It is evident that the covariant curve

$$C \equiv C_{32} + 2I_3 C_{22} = 0 \tag{4.2}$$

cuts the Neuberg curve C_{12} at infinity which is one of the four special points under consideration. The same covariant must therefore pick out of C_{12} the remaining three points t_1, t_2, t_3 of this special set. That such is the case may be easily verified. However the covariant has eight points in common with C_{12} so that the original tetrad determines a second tetrad. Writing C_{12} and the covariant C as quadratics in $\bar{x}$ and calculating their eliminant we have the eight common points of C_{12} and C given as roots of the following equation of the eighth degree:

$$\begin{aligned} &x^7 - 2\rho i x^6 + 2\rho i s_3 x^5 + (8\rho^2+7)s_3 x^4 - 4s_3(6\rho i - \rho^2 s_3)x^3 \\ &\quad - 4s_3(5\rho i s_3 + 4\rho^2 - 2\rho^3 i s_3)x^2 - 8{s_3}^2(2\rho^2 + 1 - \rho^3 i s_3)x \\ &\quad + 4{s_3}^2(2\rho i + \rho^2 s_3) = 0, \end{aligned}$$

where the leading coefficient is zero since infinity is a root. Knowing t_1, t_2, t_3 to be roots, the above eliminant contains the factor

$$x^3 - 2\rho i x^2 - 2\rho i s_3 x - s_3.$$

The remaining factor

$$x^4 + 4\rho i s_3 x^2 + 8 s_3 x - s_3(8\rho i + 4\rho^2 s_3)$$

has as roots the second tetrad. This tetrad consists of the incenter e_0 and the three centers of the escribed circles, e_1, e_2, e_3, of the triangle $t_1 t_2 t_3$. In terms of the t's we have

$$\begin{aligned} e_0 &= \tau_1 - \tau_2 + \tau_3, & e_1 &= \tau_1 + \tau_2 - \tau_3, \\ e_2 &= -\tau_1 - \tau_2 - \tau_3, & e_3 &= -\tau_1 + \tau_2 + \tau_3, \end{aligned}$$

where $\tau_i = (s_3/t_i)^{1/2}$.

We object to (4.2) on the ground that it is not homogeneous in the coefficients of C_{12}. An homogeneous relation is obtained as follows: It may easily be verified that the following non-homogeneous relation exists among the invariants of C_{12}, viz.

$$I_4 I_3^2 - 6I_2 I_3^2 + 9I_3^2 - I_4 + I_2^2 = 0. \tag{4.3}$$

By inserting proper powers of $-2I_3 C_{22}/C_{32}$ which is of degree two in the coefficients and by (4.2) is equal to 1, the relation (4.3) may be made homogeneous. We have then, after simplifying,

$$I_3^2\{I_4 C^3_{32} + 12 I_2 I_3 C^2_{32} C_{22} + 36 I_3^2 C_{32} C^2_{22} + 8 I_3 (I_4 - I_2^2) C^3_{22}\} = 0.$$

The part included in brackets is of degree 13 in the coefficients of C_{12} and of degree 6 in each of the variables. This covariant may cut out of C_{12} twenty-four points in three sets of two tetrads each, for it breaks up into three biquadratic factors of the type of (4.2). Whether these three sets are all real, or one real and two complex is yet to be determined.

5. *Invariants of the bicubic.* In conclusion we record the results obtained by applying the method of Section 1 to the bicubic. We have the bicubic

C_{13}	1	$3\bar{x}$	$3\bar{x}^2$	$\bar{x}^3$
1	a_{00}	a_{01}	a_{02}	a_{03}
$3\bar{x}$	a_{10}	a_{11}	a_{12}	a_{13}
$3\bar{x}^2$	a_{20}	a_{21}	a_{22}	a_{23}
$\bar{x}^3$	a_{30}	a_{31}	a_{32}	a_{33}

and its reciprocal form

C_{33}	1	x	x^2	x^3
1	A_{33}	A_{32}	A_{31}	A_{30}
$\bar{x}$	A_{23}	A_{22}	A_{21}	A_{20}
$\bar{x}^2$	A_{13}	A_{12}	A_{11}	A_{10}
$\bar{x}^3$	A_{03}	A_{02}	A_{01}	A_{00}.

The expansion of the determinant of the pencil

$$C_{13} + kC_{33},$$

as to powers of k, we find to be

$$I_4^3 k^4 + 2I_4^2 I_2 k^3 + I_4(18I_4 + I_2^2)k^2 + 18 I_4 I_2 k + 81 I_4,$$

where the invariant I_4 is the determinant of the coefficients of C_{13} and

$$I_2 = (a_{00}a_{33} - a_{03}a_{30}) - 3(a_{01}a_{32} - a_{02}a_{31}) - 3(a_{10}a_{23} - a_{13}a_{20}) + 9(a_{11}a_{22} - a_{12}a_{21}).$$

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An Extension of Brahmagupta's Theorem.

By A. W. Richeson.

The question of cyclic polygons early attracted the attention of mathematicians. As early as 625 A. D. we find Brahmagupta, a Hindoo mathematician, working with the cyclic quadrangle.* About 1782 Lhuilier gave a formula for the radius of the circumcircle of an inscribed quadrangle in terms of the four sides, for both the convex and the crossed cases.† It is the purpose of this paper to study first the inscribed quadrangle and then to consider other polygons, and in particular the inscribed pentagon.

1. *Distances under Inversions.* Two points P and P' collinear with a given point O are said to be inverse points with respect to the point O if the product

$$\overline{OP} \cdot \overline{OP'} = K$$

where K is a real constant, O is the center, and K the constant of inversion. Given the inverse points $P_0, P_1, P_2 \cdots$ in the plane, let us denote by λ_{ij} the absolute distances of the points P_i and P_j. Under an inversion with respect to the point O as a center the distance λ_{12} becomes

$$K\lambda'_{12}/\lambda'_{01}\,\lambda'_{02}.$$

Accordingly a function of the distances which contains each suffix the same number of times merely acquires a factor under an inversion, that is to say, it is a relative invariant. Consequently for four points any one of the eight expressions

$$\lambda^2_{01}\lambda_{23}\lambda_{34}\lambda_{42} \pm \lambda^2_{02}\lambda_{34}\lambda_{41}\lambda_{13} \pm \lambda^2_{03}\lambda_{41}\lambda_{12}\lambda_{24} \pm \lambda^2_{04}\lambda_{12}\lambda_{23}\lambda_{31}$$

is a relative invariant under inversions.

Since equations of the type

$$\mu_1\overline{OP}_1{}^2 + \mu_2\overline{OP}_2{}^2 + \mu_3\overline{OP}_3{}^2 + \cdots + K = 0$$

are equations of circles, then

$$\lambda^2_{p1}\lambda_{23}\lambda_{34}\lambda_{42} \pm \lambda^2_{p2}\lambda_{34}\lambda_{41}\lambda_{13} \pm \lambda^2_{p3}\lambda_{41}\lambda_{12}\lambda_{24} \pm \lambda^2_{p4}\lambda_{12}\lambda_{23}\lambda_{31} = 0,$$

* Kaye, G. R., *Indian Mathematics*, *Thacker Spink Co.*, Calcutta (1915), p. 22.

† Archibald, R. C., "Area of the Quadrilateral," *Bulletin of the Mathematical Association of America*, Vol. 29, p. 32.

where p denotes the variable point and the λ's the absolute distances from the points, is the equation of a circle.

Attached to four points in the plane there is a system of eight circles obtained by considering the arrangement of the positive and the negative signs in

$$\sum^{4} \lambda^2_{p1}\,\lambda_{23}\,\lambda_{34}\,\lambda_{42} \pm = 0.$$

This system of circles may be classified as follows:

(1) When all the signs are positive, one nullipartite circle. This is the standard case.

(2) When two signs are positive and two negative, three unipartite circles, the Jacobian circles of the four points.

(3) When one sign is positive and three negative, four null or point circles.

In a paper in the *Proceedings of the London Mathematical Society,* R. Russell * considers seven of this system, the three unipartite or Jacobian circles and the four null or point circles. However, he failed to take into consideration the nullipartite circle.

I propose to determine the radii of this system of circles, and to consider what happens when the four points are on a circle.

2. *The Standard Case.* The equation of the circle in the standard case is

$$\lambda^2_{p1}\,\lambda_{23}\,\lambda_{34}\,\lambda_{42} + \lambda^2_{p2}\,\lambda_{34}\,\lambda_{41}\,\lambda_{13} + \lambda^2_{p3}\,\lambda_{41}\,\lambda_{12}\,\lambda_{24} + \lambda^2_{p4}\,\lambda_{12}\,\lambda_{23}\,\lambda_{31} = 0$$

where the λ's denote the absolute distances from the points and p is the variable point. In order to calculate the radius it is convenient to make a slight change in notation. Let x be the variable point p,

$$\begin{aligned} c &= \lambda_{12} & e &= \lambda_{24} \\ b &= \lambda_{13} & f &= \lambda_{34} \\ a &= \lambda_{23} & d &= \lambda_{41} \end{aligned}$$

Then the expression for the circle becomes

$$|\,x - x_1\,|^2 aef + |\,x - x_2\,|^2 bdf + |\,x - x_3\,|^2 cde + |\,x - x_4\,|^2 abc = 0$$

which may be written

$$(1)\quad (x - x_1)(\bar{x} - \bar{x}_1)aef + (x - x_2)(\bar{x} - \bar{x}_2)bdf + (x - x_3)(\bar{x} - \bar{x}_3)cde + (x - x_4)(\bar{x} - \bar{x}_4)abc = 0.$$

* Russell, R., "Geometry on the Quartic," *Proceedings of the London Mathematical Society,* Vol. 19 (1889), p. 56.

Expanding and collecting the terms of (1) we will be able to compute the center and radius of this circle in terms of the six distances a, b, c, d, e and f by equating the coefficients of (1) with those of the self conjugate equation of the circle

$$\alpha x\bar{x} + \bar{a}_1 x + a_1\bar{x} + \beta = 0,$$

where the center $y = -a_1/\alpha$ and the radius

$$R^2 = (a_1\bar{a}_1 - \alpha\beta)/\alpha^2.$$

In this way we have the center

$$y = (-a_1)/\alpha = (aefx_1 + bfdx_2 + cdex_3 + abcx_4)/(aef + bfd + cde + abc)$$

which recalls the theorem that the centroid of four points is the sum of the products of the four points and their respective weights divided by the weights of the four points.

In a similar manner the radius is found to be

$$(2) \qquad \begin{aligned} R^2 &= (a_1\bar{a}_1 - \alpha\beta)/\alpha^2 \\ &= -2abcdef(ad + cf + be)/(aef + bfd + cde + abc)^2. \end{aligned}$$

This is the radius of the standard case of the circle and applies with change of sign to any one of the system of eight circles.

3. *The Jacobian Circle.* In order to obtain one of the Jacobian circles from

$$\sum^{4} \lambda^2_{p1}\lambda_{23}\lambda_{34}\lambda_{42} \pm = 0$$

it is necessary to change the sign of one of the λ's, say λ_{13}, from positive to negative. We may then write

$$(3) \quad \lambda^2_{p1}\lambda_{23}\lambda_{34}\lambda_{42} - \lambda^2_{p2}\lambda_{34}\lambda_{41}\lambda_{13} + \lambda^2_{p3}\lambda_{41}\lambda_{12}\lambda_{24} - \lambda^2_{p4}\lambda_{12}\lambda_{23}\lambda_{31} = 0.$$

If this circle is on the point 1 then

$$-\lambda^2_{12}\lambda_{34}\lambda_{41}\lambda_{13} + \lambda^2_{13}\lambda_{41}\lambda_{12}\lambda_{24} - \lambda^2_{14}\lambda_{12}\lambda_{23}\lambda_{31} = 0.$$

This says that the point 1 is on the Jacobian circle if

$$-\lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} - \lambda_{14}\lambda_{23} = 0.$$

Likewise for the points 2, 3 and 4 the same condition holds true. That is, the Jacobian circle is the circumcircle of the four points in the above order if $-\lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} - \lambda_{14}\lambda_{23} = 0$. This is a known condition that four points be on a circle in the order 1, 2, 3, 4.

Since λ_{13} in

$$\sum^{4} \lambda_{p1}{}^2\lambda_{23}\lambda_{34}\lambda_{42} \pm = 0$$

was replaced by b and its sign changed to obtain the Jacobian circle, we must likewise change the sign of b in the expression (2) above for the radius of the standard case to obtain the radius of the Jacobian circle. Hence we may write

$$(4) \qquad R^2 = 2abcdef(ad + cf - be)/(aef - bfd + cde - abc)^2$$

for the radius of the Jacobian circle. But this value of the radius is indeterminate. For the condition that four points be on a circle is that $ad + cf - be = 0$ or that $aef - bfd + cde - abc = 0$. This is only one condition since the one implies the other.

In order to obtain the limit of this indeterminate expression (4) let us consider Cayley's relation connecting the mutual distances of any four points in the plane. In determinant * form it is as follows:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \lambda_{12}{}^2 & \lambda_{13}{}^2 & \lambda_{14}{}^2 \\ 1 & \lambda_{21}{}^2 & 0 & \lambda_{23}{}^2 & \lambda_{24}{}^2 \\ 1 & \lambda_{31}{}^2 & \lambda_{32}{}^2 & 0 & \lambda_{34}{}^2 \\ 1 & \lambda_{41}{}^2 & \lambda_{42}{}^2 & \lambda_{43}{}^2 & 0 \end{vmatrix} = 0.$$

Expanding and replacing the λ's by a, b, c, d, e and f we may write this in the form

$$\begin{aligned} &a^2d^2(d^2 - e^2 - f^2) + a^2d^2(a^2 - b^2 - c^2) + b^2e^2(e^2 - d^2 - f^2) \\ &\quad + b^2e^2(b^2 - c^2 - a^2) + c^2f^2(f^2 - e^2 - d^2) + c^2f^2(c^2 - a^2 - b^2) \\ &\quad + a^2e^2f^2 + b^2f^2d^2 + c^2d^2e^2 + a^2b^2c^2 = 0. \end{aligned}$$

Now it is possible to express the left hand side in the form

$$\begin{aligned} &(ad + cf + be)\{2ad(a^2 + d^2) + 2be(b^2 + e^2) + 2cf(c^2 + f^2) \\ &\quad -(ad + cf + be)\sum^{6} a^2\} + (aef + bfd + cde + abc)^2 \end{aligned}$$

which is of the form $g_3{}^2x + g_2y$ where $g_2 = ad + cf + be$ and $g_3 = aef + bfd + cde + abc$. From this we see that when $g_2 = 0$,

$$g_2/g_3{}^2 = -1/[2ad(a^2 + d^2) + 2be(b^2 + e^2) + 2cf(c^2 + f^2)]$$

and (2) becomes

$$R^2 = abcdef/\sum^{3} ad(a^2 + d^2).$$

Changing the sign of b, we have for the points 1, 2, 3, 4 on a circle (in this order)

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* *Cayley's Collected Works*, Vol. I, p. 3; Salmon, Conic Sections, p. 134.

$$R^2 = -abcdef/[ad(a^2+d^2) - be(b^2+e^2) + cf(c^2+f^2)].$$

The two diagonals b and e may be expressed in terms of the other four distances * as

$$b^2 = (ad+cf)(af+cd)/(ac+df)$$

and

$$e^2 = (ad+cf)(ac+df)/(af+cd).$$

When these two relations are substituted in the expression for the radius above, it readily reduces to *

(4′) $R^2 = (ac+df)(ad+cf)(af+cd)/(s-a)(s-c)(s-d)(s-f)$,

where $2s = a+c+d+f$. This gives the radius of the circumcircle of the four points in terms of the four distances a, c, d and f when the points are in the order x_1, x_2, x_3, x_4.

In order to obtain an expression for the radius of the Jacobian circle when the points are in the order x_1, x_2, x_4, x_3, that is when the quadrangle formed by the four distances a, c, d and f is crossed, it is necessary to change the sign of one of the sides, say f, in equation (4′). The radius then reduces to

$$R^2 = (ac-df)(ad-cf)(cd-af)/s(s-a-f)(s-c-f)(s-d-f),$$

where $2s = a+c+d+f$.

4. *The Inscribed Hexagon.* In the above sections the radius of the circumcircle of the quadrangle was calculated in terms of the four distances connecting the points for two cases, the convex quadrangle and the crossed quadrangle. In this and the following sections we shall consider other inscribed polygons, in particular the hexagon and the pentagon. For the hexagon we shall take the convex hexagon as the standard case.

We may think of the hexagon as made up of two quadrangles with a common side; i. e. the two quadrangles $abcx$ and $defx$ where x is the common side. From section (3) we have two equations for the circumcircle of the two quadrangles

(5) $$R^2 = \frac{(ab+cx)(ac+bx)(ax+bc)}{(a+b+c-x)(b+c+x-a)(c+x+a-b)(x+a+b-c)}$$

and

(6) $$R^2 = \frac{(de+fx)(df+ex)(dx+ef)}{(d+e+f-x)(e+f+x-d)(f+x+d-e)(x+d+e-f)}$$

These two equations must coexist if the hexagon is inscribed in the circle. We may further write the equality

* Hobson, *Plane Trigonometry* (Fifth Edition), p. 206.

$$(A)\qquad \frac{(ab+cx)(ac+bx)(ax+bc)}{(a+b+c-x)(b+c+x-a)(c+x+a-b)(x+a+b-}$$
$$= \frac{(de+fx)(df+ex)(dx+ef)}{(d+e+f-x)(e+f+x-d)(f+x+d-e)(x+d+e-}$$

This is an equation of the seventh degree in x and on eliminating x from equations (5) and (6) above we will obtain an equation of the seventh degree in R^2 of the type

$$(7)\qquad \lambda_0\alpha_0R^{14}+\lambda_1\alpha_1R^{12}+\lambda_2\alpha_2R^{10}+\lambda_3\alpha_3R^8+\lambda_4\alpha_4R^6 + \lambda_5\alpha_5R^4+\lambda_6\alpha_6R^2+\lambda_7\alpha_7=0$$

where the λ's are numerical constants and the α's are some function of the sides a, b, c, d, e, f. Simplifying the two equations and clearing of fractions we have

$$(8)\quad R^2x^4+abcx^3+[a^2b^2+a^2c^2+b^2c^2-2R^2(a^2+b^2+c^2)]x^2 + abc(a^2b^2c^2-8R^2)x+[a^2b^2c^2-2R^2(a^2b^2+a^2c^2+b^2c^2) + R^2(d^4+e^4+f^4)]=0.$$

and

$$(8')\quad R^2x^4+defx^3+[d^2e^2+d^2f^2+e^2f^2-2R^2(d^2+e^2+f^2)]x^2 + def(d^2e^2f^2-8R^2)x+[d^2e^2f^2-2R^2(d^2e^2+d^2f^2+e^2f^2) + R^2(d^4+e^4+f^4)]=0.$$

If we set

$$\sigma_3=abc \qquad s_3=def$$
$$\Sigma a^2b^2=a^2b^2+a^2c^2+b^2c^2 \qquad \Sigma d^2e^2=d^2e^2+d^2f^2+e^2f^2$$
$$\Sigma a^4=a^4+b^4+c^4 \qquad \Sigma d^4=d^4+e^4+f^4$$

equations (8) and (8') can be expressed in the condensed form

$$(9)\qquad R^2x^4+\sigma_3x^3+(\Sigma a^2b^2-2R^2\Sigma a^2)x^2 + (\sigma_3{}^2-8R^2\sigma_3)x+(\sigma_3{}^2-2R^2\Sigma a^2b^2+R^2\Sigma a^4)=0$$

and

$$(9')\qquad R^2x^4+s_3x^3+(\Sigma d^2e^2-2R^2\Sigma d^2)x^2 + (s_3{}^3-8R^2s_3)x+(s_3{}^2-2R^2\Sigma d^2e^2+R^2\Sigma d^4)=0$$

and x can be eliminated from equations (9) and (9') by Bezout's method of elimination.* This will give us a determinant of the fourth order and the seventh degree in R^2. We shall designate this determination by Δ_6:

* Salmon, *Higher Algebra*, p. 81.

$$\Delta_6 = \begin{vmatrix}
(s_3 - \sigma_3), \\
(\Sigma d^2e^2 - 2R^2\Sigma d^2 - \Sigma a^2b^2 + 2R^2\Sigma a^2), \\
(s_3^3 - 8R^2s_3 - \sigma_3^3 + 8R^2\sigma_3), \\
(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4 - \sigma_3^2 + 2R^2\Sigma a^2b^2 - R^2\Sigma a^4), \\
R^2[\Sigma d^2e^2 - 2R^2\Sigma d^2 - \Sigma a^2b^2 + 2R^2\Sigma a^2], \\
R^2[s_3^3 - 8R^2s_3 - \sigma_3^3 + 8R^2\sigma_3] \\
\qquad + \sigma_3[\Sigma d^2e^2 - 2R^2\Sigma d^2] - s_3[\Sigma a^2b^2 - 2R^2\Sigma a^2], \\
R^2(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^2 - \sigma_3^2 + 2R^2\Sigma a^2b^2 - R^2\Sigma a^4) \\
\qquad + \sigma_3(s_3^3 - 8R^2s_3) - s_3(\sigma_3^3 - 8R^2\sigma_3), \\
\sigma_3(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4) - s_3(\sigma_3^2 - 2R^2\Sigma a^2b^2 + R^2\Sigma a^4), \\
R^2(s_3^3 - 8R^2s_3 - \sigma_3^3 + 8R^2\sigma_3), \\
R^2(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4 - \sigma_3^2 + 2R^2\Sigma a^2b^2 - R^2\Sigma a^4) \\
\qquad + \sigma_3(s_3^3 - 8R^2s_3) - s_3(\sigma_3^3 - 8R^2\sigma_3), \\
\sigma_3(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4) - s_3(\sigma_3^2 - 2R^2\Sigma a^2b^2 + R^2\Sigma a^4) \\
\quad + (\Sigma a^2b^2 - 2R^2\Sigma a^2)(s_3^3 - 8R^2s_3) - (\Sigma d^2e^2 - 2R^2\Sigma d^2)(\sigma_3^3 - 8R^2s_3), \\
(\Sigma a^2b^2 - 2R^2\Sigma a^2)(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4) \\
\qquad - (\sigma_3^2 - 2R^2\Sigma a^2b^2 + R^2\Sigma a^4)(\Sigma d^2e^2 - 2R^2\Sigma d^2), \\
R^2(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4 - \sigma_3^2 + 2R^2\Sigma a^2b^2 - R^2\Sigma a^4) \\
\sigma_3(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4) - s_3(\sigma_3^2 - 2R^2\Sigma a^2b^2 + R^2\Sigma a^4) \\
(\Sigma a^2b^2 - 2R^2\Sigma a^2)(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4) \\
\qquad - (\sigma_3^2 - 2R^2\Sigma a^2b^2 + R^2\Sigma a^4)(\Sigma d^2e^2 - 2R^2\Sigma d^2) \\
(\sigma_3^3 - 8R^2\sigma_3)(s_3^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4) \\
\qquad - (s_3^3 - 8R^2s_3)(\sigma_3^2 - 2R^2\Sigma a^2b^2 + R^2\Sigma a^4)
\end{vmatrix}$$

Let us prove the following theorem:

THEOREM. *The product, $R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot \cdots R_7^2$, of the seven roots of R^2 given by equation (7) after eliminating x from equations (9) and (9′) above is*

$$K \prod^{10} (abc - def) / \prod^{16} (a \pm b \pm c \pm d \pm e \pm f)$$

where K is a numerical constant and with the agreement that there shall be an odd number of negative signs in the factors of the denominator.

Let us examine the two coefficients $\lambda_0\alpha_0$ and $\lambda_7\alpha_0$ in equation (7) above. From equations (5) and (6) R^2 can become infinite when $x = a + b + c$ and $x = d + e + f$, that is, when at least one factor of the denominators of both (5) and (6) is zero. Hence R^2 is infinite if $a + b + c = d + e + f$ or $a + b + c - d - e - f = 0$. But since there are four factors in each denominator, for each factor of one there are four factors that go with it. This is the only way in which R^2 can become infinite since R^2 is not infinite when x is infinite. Hence $\lambda_1 \alpha_0 = \lambda_0 \prod^{16} (a \pm b \pm c \pm d \pm e \pm f)$.

Now let us examine $\lambda_7\alpha_7$ in equation (7). From equations (5) and (6) R^2 can become zero if $ab + cx = 0$ and $de + fx = 0$. That is, R^2 will become zero if $abf - dec = 0$; but since there are three factors in each numerator of (5) and (6), for each factor of the one there are three factors that go with it. Hence we have the product

$$\overset{9}{\Pi}(abf - dec).$$

However on examining equation (7) in R^2, that is

$$\lambda_0 \overset{16}{\Pi}(a \pm b \pm c \pm d \pm e \pm f)R^{14} + \cdots + \lambda_7\alpha_7 = 0,$$

we find that each term should be of degree thirty, while $\overset{9}{\Pi}(abf - dec)$ is of degree twenty-seven, hence there is a missing factor. If $x = \infty$, R^2 will be zero. That is $(abc - def)$ is zero and the missing factor is found, and we have $\lambda_7\alpha_7 = \lambda_7 \overset{10}{\Pi}(abc - def)$; this is of degree thirty which is correct. Equation (7) may be written

$$\lambda_0 \overset{16}{\Pi}(a \pm b \pm c \pm d \pm e \pm f)R^{14} + \cdots + \lambda_7 \overset{10}{\Pi}(abc - def) = 0.$$

The product of the seven roots of R^2 is given by $\lambda_7\alpha_7/\lambda_0\alpha_0$. Hence we have

$$R_1^2 \cdot R_2^2 \cdot R_3^2 \cdots R_7^2 = K \overset{10}{\Pi}(abc - def) / \overset{16}{\Pi}(a \pm b \pm c \pm d \pm e \pm f)$$

and the theorem is proved.

5. *The Inscribed Pentagon.* The same argument that was applied to the hexagon in the above section may be applied to the pentagon. Instead of two quadrangles with a common side inscribed in a circle we will have one quadrangle and a triangle having a common side to form the pentagon. That is the pentagon *abcde* is formed by the quadrangle *abcx* and the triangle *dex* where x is the common side.

The radius of the circumcircle of the pentagon is given by the equations

$$(10) \quad R^2 = \frac{(ab + cx)(ac + bx)(ax + bc)}{(a + b + c - x)(b + c + x - a)(c + x + a - b)(x + a + b - c}$$

and

$$(10') \quad R^2 = \frac{d^2e^2x^2}{(d + e + x)(e + x - d)(x + d - e)(d + e - x)}$$

These two equations must coexist if the pentagon is inscribed in the circle and likewise an equality exist similar to (A) for the hexagon in the above section. This is an equation of the seventh degree in x and on eliminating x from equations (10) and (10′) we will obtain an equation of the seventh degree in R^2 of the type

$$(11) \qquad \lambda_0\alpha_0R^{14} + \lambda_1\alpha_1R^{12} + \lambda_2\alpha_2R^{10} + \lambda_3\alpha_3R^8 + \cdots + \lambda_7\alpha_7 = 0,$$

where the λ's are numerical constants and the α's are some functions of the sides a, b, c, d, e.

The process of eliminating x from the two equations (10) and (10′) is similar to that of the hexagon but since the pentagon may be obtained directly from the hexagon by making one of its sides, say f, zero, it is not necessary to go through the elimination. On making $f = 0$ the determinant Δ_6 reduces to a determinant, which we shall designate by Δ_5 of the same order and degree in R^2 as Δ_6. The determinant Δ_5 is as follows:

$$\Delta_5 = \begin{vmatrix}
-\sigma_3, & & & & \\
(\Sigma d^2e^2 - 2R^2\Sigma d^2 - \Sigma a^2b^2 + 2R^2\Sigma a^2), & & & & \\
(-\sigma_3{}^3 + 8R^2\sigma_3), & & & & \\
(-\sigma_3{}^2 - 2R^2\Sigma d^2e^2 + R^2\Sigma d^4 + 2R^2\Sigma a^2b^2 - R^2\Sigma a^4), & & & & \\
R^2(\Sigma d^2e^2 - 2R^2\Sigma d^2 - \Sigma a^2b^2 + 2R^2\Sigma a^2), & & & & \\
R^2(-\sigma_3{}^2 + 8R^2\sigma_3) + \sigma_3(\Sigma d^2e^2 - 2R^2\Sigma d^2), & & & & \\
R^2(-2R^2\Sigma d^2e^2 + R^2\Sigma d^4 - \sigma_3{}^2 + 2R^2\Sigma a^2b^2 - R^2\Sigma a^4), & & & & \\
\sigma_3(-2R^2\Sigma d^2e^2 + R^2\Sigma d^4), & & & & \\
R^2(-\sigma_3{}^3 + 8R^2\sigma_3), & & & & \\
R^2(-2R^2\Sigma d^2e^2 + R^2\Sigma d^4 - \sigma_3{}^2 + 2R^2\Sigma a^2b^2 - R^2\Sigma a^4), & & & & \\
\sigma_3(-2R^2\Sigma d^2e^2 + R^2\Sigma d^4) - (\Sigma d^2e^2 - 2R^2\Sigma d^2)(\sigma_3{}^3 - 8R^2\sigma_3), & & & & \\
(\Sigma a^2b^2 - 2R^2\Sigma a^2)(-2R^2\Sigma d^2e^2 + R^2\Sigma d^4) & & & & \\
\qquad -(\sigma_3{}^2 - 2R^2\Sigma a^2b^2 + R^2\Sigma a^4)(\Sigma d^2e^2 - 2R^2\Sigma d^2), & & & & \\
R^2(-R^2\Sigma d^2e^2 + R^2\Sigma d^4 - \sigma_3{}^2 + 2R^2\Sigma a^2b^2 - R^2\Sigma a^4) & & & & \\
\sigma_3(-2R^2\Sigma d^2e^2 + R^2\Sigma d^4) & & & & \\
(\Sigma a^2b^2 - 2R^2\Sigma a^2)(-2R^2\Sigma d^2e^2 + R^2\Sigma d^4) & & & & \\
\qquad -(\sigma_3{}^2 - 2R^2\Sigma a^2b^2 + R^2\Sigma a^4)(\Sigma d^2e^2 - 2R^2d^2) & & & & \\
(\sigma_3{}^3 - 8R^2\sigma_3)(-2R^2\Sigma d^2e^2 + R^2\Sigma d^4) & & & &
\end{vmatrix}$$

Let us prove a corresponding theorem for the pentagon as was proved for the hexagon.

THEOREM. *The product, $R_1{}^2 \cdot R_2{}^2 \cdot R_3{}^2 \cdots R_7{}^2$, of the seven roots of R^2, given by equations* (11) *after eliminating x from equations* (10) *and* (10′), *is*

$$K(abcde)^6 / \prod^{16} (a \pm b \pm c \pm d \pm e),$$

where K is a numerical constant.

We shall examine the two coefficients $\lambda_0\alpha_0$ and $\lambda_7\alpha_7$ in equation (11). From equations (10) and (10′), R^2 can become infinite when $x = a + b + c$ and $x = d + e$; that is, when at least one of the factors of the denominators

of (10) and (10′) are zero. Hence $a+b+c-d-e=0$. But since there are four factors in each denominator, for each factor of the one there are four factors that go with it. This is the only way in which R^2 can become infinite since R^2 is not infinite when x is infinite. Hence

$$\lambda_0\alpha_0=\lambda_0\prod^{16}(a\pm b\pm c\pm d\pm e).$$

Now let us examine $\lambda_7\alpha_7$. From equations (10) and (10′) R^2 can become zero if $ab+cx=0$ and either $d=0$ or $e=0$. That is to say $\lambda_7\alpha_7=\lambda_7(abcde)^\mu$, where μ must be determined by examining equation (11) in R^2. We find that each term is of degree thirty and μ must be 6. Equation (11) may be written

$$\lambda_0\prod^{16}(a\pm b\pm c\pm d\pm e)R^{14}+\cdots+\lambda_7(abcde)^6=0.$$

Since the product of the seven roots of R^2 is given by $\lambda_7\alpha_7/\lambda_0\alpha_0$ we have

$$R_1^2\cdot R_2^2\cdot R_3^2\cdots R_7^2=K(abcde)^6/\prod^{16}(a\pm b\pm c\pm d\pm e).$$

To determine the constant K we will consider the special case of the regular convex pentagon where $a=b=c=d=e$. The equations giving the radius of the circumcircle are

$$R^2=a^3(x+a)^3/(x+a)^3(3a-x)=a^3/(3a-x) \tag{12}$$

or $x=-a$ and

$$R^2=a^4x^2/(2a+x)(2a-x)x^2=a^4/(4a^2-x) \tag{12′}$$

or $x=0$. From (12) above

$$x=(3aR^2-a^3)/R^2.$$

Substituting this value of x in (12′) we obtain an equation of the second degree in R^2

$$5R^4-5a^2R^2+a^4=0.$$

$$\therefore\quad R^2=[5\pm(5)^{\frac{1}{2}}a^2]/10. \tag{13}$$

It should be noticed above that for $x=0$ in (12′) that

$$R^2=a^3/3 \tag{13′}$$

five times, since x was taken out of the two equations (12) and (12′) five times as a factor. Equation (13) gives two values of R^2 while (13′) gives five values which make up the total of seven values as shown by the general equation in R^2.

To determine the constant K required in the expression for the product of the seven roots of R^2 we have only to multiply these seven values together

$$R_1^2\cdot R_2^2\cdot R_3^2\cdots R_7^2=(1/5)(1/3)^5a^{14}.$$

Hence $K=(1/5)(1/3)^5$ and the expression for the product of the seven roots of R^2 is

$$(1/5)(1/3)^5\ (abcde)^6/\prod^{16}(a\pm b\pm c\pm d\pm e),$$

and the theorem is proved.

6. *The equation of the Seventh Degree in R^2.* To consider the general case of the pentagon where no two of the sides are equal we will center our attention on the determinant $\Delta_5=0$ obtained by eliminating x from equations (10) and (10′) in sec. 5 above. Upon expansion this determinant gives an equation of the type

$$\lambda_0\alpha_0R^{14}+\lambda_1\alpha_1R^{12}+\lambda_2\alpha_2R^{10}+\lambda_3\alpha_3R^8+\lambda_4\alpha_4R^6+\lambda_5\alpha_5R^4+\lambda_6\alpha_6R^2+\lambda_7\alpha_7=0, \tag{14}$$

where the λ's are constants and the α's are some function of the sides a, b, c, d, e. Instead of expanding $\Delta_5=0$ in full let us examine several of the coefficients of equation (14). We have already found in sec. 5 that

$$\lambda_0\alpha_0=\lambda_0\prod^{16}(a\pm b\pm c\pm d\pm e)\quad\text{and}\quad\lambda_7\alpha_7=(abcde)^6.$$

The coefficients α_4, α_5, α_6 can be obtained in terms of the symmetric functions of the squares of the sides a, b, c, d, e of the pentagon by a partial expansion of Δ_5. Thus, if we set

$$S_1=\sum^{5}a^2,\ S_2=\sum^{10}a^2b^2,\ S_3=\sum^{10}a^2b^2c^2,\ S_4=\sum^{5}a^2b^2c^2d^2,\ S_5=a^2b^2c^2d^2e^2,$$

these coefficients are as follows:

$$\alpha_7=S_5^3,\ \alpha_6=S_5^2(2S_4+2S_1S_3+2S_2^2),\ \alpha_5=S_5(4S_5S_2S_1+S_4S_3S_1-4S_4^2),$$
$$\alpha_4=2S_5^2S_2+S_5S_4S_3+S_3^4+2S_4^2S_1^4-S_4^2S_2^2-4S_4^3-2S_3^2S_4S_1^2.$$

Substituting these values for the α's we may write equation (14) as

$$\prod^{16}(a\pm b\pm c\pm d\pm e)R^{14}+\cdots$$
$$+(2S_5^2S_2+S_5S_4S_3+S_3^4+2S_4^2S_1^4-S_4^2S_2^2-4S_4^3-2S_3^2S_4S_1^2)R^6$$
$$+S_5(4S_5S_2S_1+S_4S_3S_1-4S_4^2)R^4$$
$$+S_5^2(2S_4+2S_1S_3+2S_2^2)R^2+S_5^3=0.$$

If one of the sides, say e, of the pentagon is made zero then S_5 which is the product of the squares of the five sides becomes zero. Hence each of the coefficients α_7, α_6 and α_5 becomes zero and α_4, that is the coefficient of R^6, reduces to

$$\{\prod^{3}(a^2b^2-c^2d^2)\}^2.$$

The remaining coefficients α_2 and α_3 are known in terms of S_1, S_2, S_3 and S_4 modulo S_5. The general equation of the pentagon reduces to

$$\overset{16}{\Pi}(a \pm b \pm c \pm d)R^{14} + \cdots \{\overset{3}{\Pi}(a^2b^2 - c^2d^2)\}^2R^6 = 0,$$

which is

$$[\{\overset{4}{\Pi}(a \pm b \pm c \pm d)R^2 - \overset{3}{\Pi}(ab + cd)\}\{\overset{4}{\Pi}(a \pm b \pm c \pm d)R^2 - \overset{3}{\Pi}(ab - cd)\}]$$

The first factor of this product when set equal to zero is the equation for the radius of the circumcircle of a convex quadrangle with sides a, b, c, d and the second factor when set equal to zero is likewise the equation for the radius of the circumcircle of a crossed quadrangle with the same sides a, b, c, d.

7. *The Seven Circumcircles.* We have already seen that the general equation for the pentagon obtained by eliminating x from equations (10) and (10′) is of the seventh degree in R^2. That is to say with five rods a, b, c, d, e, which when jointed form a convex pentagon inscribed in a circle we may be able to obtain six other pentagons by forming all possible crossed pentagons with the same five rods. To obtain the seven pentagons and their circumcircles it is assumed that the five rods a, b, c, d, e are of lengths a, $a+\alpha$, $a+\beta$, $a+\gamma$, $a+\delta$ respectively where α, β, γ and δ are small compared with a and $\alpha \neq \beta \neq \gamma \neq \delta$. Let us consider some facts concerning these pentagons and their circumcircles. These circles are drawn with the sides of the pentagons in their order of length and we obtain the convex pentagon, two pentagons with one crossing similar to Fig. 1, three pentagons with two crossings similar to Fig. 2, and one with five crossings similar to Fig. 3. Of the seven circumcircles no two of them have equal radii. Further if we change the order of the sides we will obtain the same number of pentagons inscribed in the same circles with the same radii $R_1, R_2, \cdots, R_7$. However, there may be a difference in the number of pentagons with one and two crossings. It is also evident that the maximum number of crossings for the pentagon is five while it is impossible to obtain three or four crossings.

8. *The Inscribed* $(2n+1)$*-gon.* Let us consider the general case of the inscribed polygon of $(2n+1)$ sides. The $(2n+1)$-gon may be built up of two polygons of $(n+1)$ and $(n+2)$ sides having one side, say x, in common. Then two equations for the radius of the circumcircle of the type of (10) and (10′) of sec. 5 can be written which must coexist if the polygons are inscribed in a circle. Eliminating x the common side from these two equations we obtain an equation in R^2 of degree

$$\tfrac{1}{2}(2n+1)\binom{2n}{n} - 2^{2n-1}.*$$

* Dr. F. Morley, Lectures at the Johns Hopkins University, 1927-28.

This expression gives the number of polygons formed by $(2n+1)$ jointed rods when the $(2n+1)$ jointed rods for the convex case form an inscribed $(2n+1)$-gon. Equally well this expression gives the number of circles with radii $R_1, R_2, \cdots, R_{2n+1}$ in which these polygons are inscribed. Further it is easy to show that the series

$$\text{(L)} \qquad n+(n-1)\binom{2n+1}{1}+(n-2)\binom{2n+2}{2}\cdots$$

has exactly this expression for its sum.

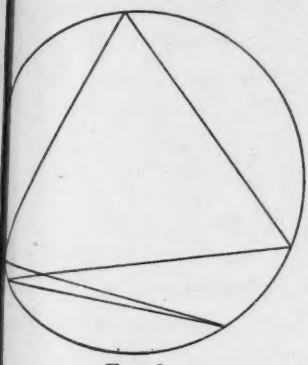

FIG. 1.

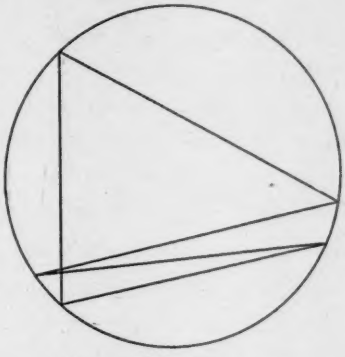

FIG. 2.

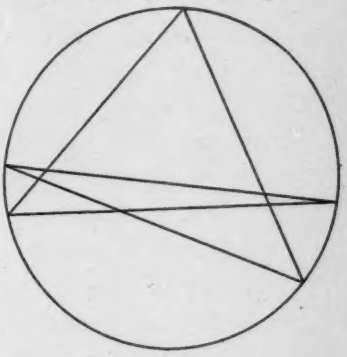

FIG. 3.

Let us examine a few special cases, say when $n=2, 3, 4$; i. e. when $2n+1=5, 7, 9$. Series (L) gives for these cases the following:

(M) $\qquad n=2, \quad 2n+1=5, \quad 2+5=7;$

(N) $\qquad n=3, \quad 2n+1=7, \quad 3+2(7)+21=38;$

(O) $\qquad n=4, \quad 2n+1=9, \quad 4+3(9)+2(36)+84=187.$

Expressions (M), (N) and (O) give the number of polygons formed by five, seven and nine jointed rods. Equally well these expressions give the number of circles in which these polygons are inscribed. In a similar manner we might extend this to $(2n+1)$ jointed rods.

Let us consider the numbers given by the last term in the left hand side of the expressions (M), (N) and (O), that is the numbers 5, 21 and 84. These are the number of times that each of the $(2n+1)$-gons degenerate into an equilateral triangle when all the sides are equal. In sec. 5 above we saw that in the case of the regular pentagon we obtained two inscribed pentagons, the regular convex and the regular star pentagon, and the degenerate case of the equilateral triangle five times over. For the heptagon the number of degenerate cases is 21, while for the 9-gon it is 84.

Let us examine each of these cases and see how this comes about. The case of the pentagon resolves itself into this: with five rods of equal length

labelled 1, 2, 3, 4, 5 forming a regular convex pentagon, how many distinct equilateral triangles can we form with these five rods? The answer is five. This may be accomplished as follows: bring sides 1 and 2 of the pentagon together until an angle of 60° is formed, these two sides are two of the sides, A and B, of the equilateral triangle. Now fold sides 3, 4 and 5 of the pentagon together forming the third side C of the triangle. This will give one of the triangles. Continue this procedure by taking two adjacent sides of the pentagon to form two sides of the triangle and in this manner we will form five distinct triangles as shown below.

Sides of triangle	A	B	C
Sides of pentagon	1	2	345
	2	3	451
	3	4	512
	4	5	123
	5	1	234

In the above we took the regular convex pentagon but the same results would have been obtained with the regular crossed or star pentagon.

For the heptagon we have the same problem. With seven rods of equal length labelled 1, 2, 3, 4, 5, 6, 7 how many distinct equilateral triangles can be formed? The answer is 21; this agrees with the number given by expression (N) above. They are as follows:

A	B	C
1	2	34567
.	.	.
7	1	23456
123	456	7
.	.	.
712	345	6
1234	56	7
.	.	.
7123	45	6

There are seven distinct triangles in each group which makes a total of twenty-one.

In a similar manner we can obtain eighty-four distinct equilateral triangles with nine labelled rods of equal length. We might extend this process on to polygons of $(2n+1)$ equal sides where we would obtain $\binom{2n+1}{n-1}$ equilateral triangles.